Darmstadt - Frankfurt Seminar

Toric varieties and modular forms

Sommersemester 2017

Organizers: Yingkun Li (Darmstadt), Alejandro Soto (Frankfurt)

The main objective of this seminar is the study of toric modular forms, as introduced by Borisov and Gunells in [2], and its applications [1], [3] and [5]. For this, we start with the basic theory of toric varieties and modular forms, before moving to the main definitions and properties. In particular, we will study the cohomology of toric varieties as it plays an important role in the theory. At the end we discuss some applications constructing equations of modular curves and Hilbert modular varieties.

In Frankfurt the seminar takes place at 15:00 in room 711 (Groß), at Robert–Mayer–Straße 10.
In Darmstadt, it takes place at 15:15 in room S2|15 room 234 at Schloßgartenstraße 7.

1 27.04.2017 Darmstadt

Talk 1.1: Introduction. (Yingkun Li)
General overview and objectives of the seminar.

Talk 1.2: Crash course in modular forms. (Johannes Buck)
Give a brief introduction to important concepts in modular forms. Discuss congruence subgroups $\Gamma_0(N)$ and $\Gamma_1(N)$. Introduce the modular curves $X_0(N)$ and $X_1(N)$ as compactified quotients of the upper half plane, and modular forms as sections of

line bundles. State the Riemann-Roch theorem and deduce from it the dimension formula of $M_k(\Gamma)$ for congruence subgroup Γ and even k [8, §3.4-3.5]. Give two examples of modular forms:

- Jacobi theta function $\vartheta(z, \tau)$. Prove the Jacobi triple product formula, the modular transformation formula, and that it satisfies the heat equation.
- Integral weight Eisenstein series for $\Gamma_0(N)$ [8, §4.2]. For $\mathrm{SL}_2(\mathbb{Z})$ and weight $k \geq 4$, derive its Fourier expansion. Mention the Fourier expansion for weight 2 and weight $k \geq 3$ of level N.

Introduce Hecke operators and their actions on the Fourier expansion [8, §5.2-5.3]. Discuss Atkin-Lehner operators and Fricke involutions for $\Gamma_1(N)$ and their actions on cusps. Discuss newform, eigenform, its Mellin transform and L-function [8, §5.6, 5.8, 5.9]. Talk about the Petersson inner product and the decomposition of $M_k(\Gamma)$ into direct sum of Eisenstein series and cusp forms.
Introduce the Hilbert modular group (of level 1) for a totally real field F of degree n, its action on n copies of the upper-half plane. Introduce Hilbert modular forms, discuss the Koecher principle (if time permits, give a proof). Finally, state the Fourier expansion of Hilbert Eisenstein series of level 1 [10, §1.1, 1.2, 1.4, 1.5], [6, §1.3].

Background: complex analysis, basic algebraic geometry.
Literature: [6], [8], [10].

2 11.05.2017 Frankfurt

Talk 2.1: Toric Varieties 1. (Rosemarie Martienssen)
Begin with the definition and properties of toric varieties, giving along the way some examples illustrating the theory [11, Chapter I, §1–2], [7, Chapter 1–3]. Define rational polyhedral cones and produce affine toric variety from it [7, Theorem 1.2.18]. Discuss morphisms between affine toric varieties and Example 1.3.17 in [7]. Define projective toric variety and produce it from lattice polytopes [7, §2.1-2.3]. Discuss the normal fan and give an example [7, Theorem 2.3.2, Example 2.3.4]. Discuss properties of affine/projective toric varieties, such as normality and smoothness [7, Theorems 1.3.12, 2.4.1, 2.4.3]. Introduce the notion of a fan, the toric variety associate to it [7, Theorem 3.1.5] and give examples.

Background: basic knowledge on algebraic varieties.
Literature: [11, Chapter I, §1–2], [7, Chapter 1–2, §3.1], [9].

Talk 2.2: Toric Varietes 2. (Max Bieri)
In this talk we continue with the study of toric varieties. In particular we will study divisors, line bundles and vector bundles on them [7, §4.1-4.3, Chapter 6]. State the Euler exact sequence and the canonical divisor of a smooth toric varieties [7, Theorems 8.1.6, 8.2.3] and give some examples. Introduce Čech cohomology for toric varieties [7, §9.1] and give Example 9.1.1.

Background: basic knowledge on toric varieties and sheaf theory.
Literature: [7, §3.1, 4.1–4.3, 8.1–8.2, 9.1].

3 01.06.2017 Darmstadt

Talk 3.1: Toric Varietes 3. (Nithi Rungtanapirom)
The aim of this talk is to explain the Hirzebruch–Riemann–Roch theorem (HRR) for vector bundles on toric varieties [7, Chapter 13]. Recall the Chern class, Chern character and Todd class of a complete variety. State the Hirzebruch-Riemann-Roch (HRR) Theorem and give the example for a smooth, complete surface [7, Example 13.1.7]. Prove HRR for smooth complete toric varieties following [13].

Background: basic knowledge on toric varieties and cohomology.
Literature: [7, Chapter 12–13], [13].

Talk 3.2: Toric modular forms 1. (Anna von Pippich)
Define toric modular forms $f_{N,\deg}$ and give Example 2.4 in [2]. Provide the homological interpretation and use it to show the convergence of the formal q-series [2, Lemma 2.9 and Theorem 2.10].

Background: experience with homological calculations a plus.
Literature: [2, §1–2].

4 08.06.2017 Frankfurt

Talk 4.1: Toric modular forms 2. (Martin Möller)
Relate toric modular form to Euler characteristic of a graded infinite dimensional bundle W. Use the Hirzebruch–Riemann–Roch Theorem and Čech complex to prove

that it is a modular form meromorphic on the upper-half plane [2, Theorems 3.4, 3.5, Prop. 4.3].

Background: experience with cohomology calculations a plus.
Literature: [2, §2–4.3].

Talk 4.2: Toric modular forms 3. (Moritz Dittmann)
Introduce the modular forms $s_{a/\ell}^{(k)}$ and $r^{(k)}$, and prove that they span the space spanned by level-ℓ toric modular forms [2, Theorem 4.11].

Background: modular forms.
Literature: [2, §4.4-4.14].

5 22.06.2017 Darmstadt

Talk 5.1: Span of toric modular forms. (Sven Möller)
Show that the space of toric modular forms is stable under Hecke operator. Introduce Manin symbols. Prove that toric modular forms of weight $k = 2$ span the space of modular forms of analytic rank 0. If time permits, discuss the analogous result for higher weight [3].

Background: experience with Hecke operators and modular symbols a plus.
Literature: [2, §4.4–5], [1] and if time permits [3].

Talk 5.2: Equation of modular curves. (Alejandro Soto)
Use toric Eisenstein series to give an embedding of the modular curve $X_1(p)$ into projective space following [5].

Background: modular forms and basic knowledge on algebraic varieties.
Literature: [5].

6 13.07.2017 Frankfurt

Talk 6.1: Toroidal Embeddings and compactifications of Hilbert modular surfaces. (Ana Maria Botero)
The aim of this talk is to generalize the constructions of toric varieties to toroidal embeddings. Explain how to construct a conical polyhedral complex from a toroidal embedding, i.e. [11, Chapter 2, Theorem 6*]. Then give a brief presentation of a

Hilbert modular surface, as done in [14, Chapter 1], and of its compactification, see [14, Chaper 2, §7] and [12, §11].

Background: basic knowledge on toric varieties.
Literature: [11, Chapter 2, §1 and §2], [14] and [12].

Talk 6.2: Model of Hilbert modular variety. (Jakob Stix)
Construct the resolutions X_{sm} and X_{ch} of the minimal compactification X of an open Hilbert modular threefold X° [4, Section 3]. Follow [15] and [4, Section 5] to introduce certain weight $(1, 1, 1)$ Hilbert Eisenstein series. Then use these Eisenstein series to show that X_{ch} is the canonical model of X [4, Theorem 8.4].

Background: experience with toroidal compactification.
Literature: [5], [4, §2–8] and [15].

References

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