THE AREA IS A GOOD ENOUGH METRIC

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Abstract. In the first part we extend the construction of the smooth normal-crossing divisors compactification of projectivized strata of abelian differentials given by Bainbridge, Chen, Gendron, Grushevsky and Möller to the case of k-differentials. Since the generalized construction is closely related to the original one, we mainly survey their results and justify the details that need to be adapted in the more general context.

In the second part we show that the flat area provides a canonical hermitian metric on the tautological bundle over the projectivized strata of finite area k-differentials whose curvature form represents the first Chern class. This result is useful in order to apply Chern-Weyl theory tools. It has already been used as an assumption in the work of Sauvaget for abelian differentials and will be used in a forthcoming paper of Chen, Möller and Sauvaget for quadratic differentials.

Contents

1. Introduction

A natural invariant of a flat surface (X, ω) is the flat area $\mathrm{vol}(X, \omega)$, the area taken with respect to the form $|\omega|$. As such, it defines a hermitian metric h on the tautological line bundle $\mathcal{O}(-1)$ over the projectivized strata $\mathbb{P}\Omega\mathcal{M}_g(\mu)$. This metric does not extend smoothly over the boundary, as the area of a flat surface tends to ∞ when X acquires an infinite flat cylinder, i.e. when ω acquires a simple pole. In Chern-Weyl theory applications, it suffices to show that the curvature form of the metric connection associated to the metric h represents the first Chern class of $\mathcal{O}(-1)$ on a suitable compactification. This has been used as assumption by Sauvaget in [Sau18] for Masur-Veech volumes of the minimal strata of abelian differentials. While a workaround for this has been given in [CMSZ19], the computation of the volume of individual spin components in loc. cit. is still based on that assumption. Moreover, the paper [CMS19] extends this line of thought to quadratic differentials. There, too, the volume of the canonical double cover (see Section 2)

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provides a natural hermitian metric. Even for principal strata, where the Hodge bundle provides a smooth compactification, we do not see an easy route to prove the claim in the title, see the subtleties explained below. This paper consequently makes full use of the smooth compactification of strata of abelian differentials constructed in [BCGGM3]. Yet another application is a growth justification in the recent computation of the volume of moduli spaces of flat surfaces (in the sense of Veech ([Vee93]) by Sauvaget [Sau20]).

Given the applications in mind, the first part of this paper is a survey about the construction of the smooth compactification and the formal justification of the tempting claim that the construction extends to k-differentials, if the notions are appropriately adapted in the same way as [BCGGM2] adapts [BCGGM1].

The compactification. Let $\mu = (m_1, \ldots, m_n)$ be a type of a meromorphic k-differential, i.e. m_i are integers such that $\sum m_i = k(2g-2)$. We summarize the properties of our compactification $\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ of the moduli space of k-differentials, the n points being labeled throughout. The construction starts with the space $\Omega^k_+\mathcal{M}_{g,n}(\mu)$ parameterizing k-differentials q plus the choice of a k-th root ω of the pullback of q to the canonical k-cover. The forgetful map makes $\Omega^k_+\mathcal{M}_{g,n}(\mu) \to \Omega^k\mathcal{M}_{g,n}(\mu)$ into an unramified cover of degree k.

Theorem 1.1. *There exists a complex orbifold $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$, the moduli space of multi-scale k-differentials, with the following properties.*

i) *The space $\Omega^k_+\mathcal{M}_{g,n}(\mu)$ is dense in $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$*
ii) *The boundary $D = \Xi^k\overline{\mathcal{M}}_{g,n}(\mu) \smallsetminus \Omega^k_+\mathcal{M}_{g,n}(\mu)$ is a normal crossing divisor.*
iii) *The rescaling action of $\mathbb{C}^*$ on $\Omega^k_+\mathcal{M}_{g,n}$ extends to $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ and the resulting projectivization $\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ is a compactification of $\mathbb{P}\Omega^k\mathcal{M}_{g,n}(\mu)$.*
iv) *Via the canonical cover construction, the space $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ is embedded as suborbifold in the compactification $\Xi\overline{\mathcal{M}}_{\widehat{g},\{\widehat{n}\}}(\widehat{\mu})$ of the corresponding stratum of abelian differentials with partially labeled points.*

Here we only prove that $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ is a 'moduli space' in a very weak form, namely by exhibiting what its complex points correspond to, the multi-scale k-differentials introduced below. We leave it to the interested reader to adapt the functor from [BCGGM3] to the context of k-differentials.

Besides the normal crossing boundary, the most relevant property for us is the existence of a convenient coordinate system, given by *perturbed period coordinates*. To introduce this, we first have to explain how to parameterize boundary points of $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$.

Let $\Gamma = (V, H, E, g)$ be a stable graph, where g is the genus assignment. A *level graph* is a stable graph together with a weak total order on the set of vertices. We usually specify the order using a level function, usually normalized to take values in $0, -1, \ldots, -L$, with zero the top level. An *enhanced level graph* is a level graph together with an enhancement $\kappa\colon H \to \mathbb{Z}$ on the half-edges that specifies the number of prongs of the differential at the corresponding marked point, see Section 2.

Each of the levels of Γ thus specifies a moduli space of k-differentials, the type being given by the enhancement. A collection of these differentials is called *twisted differential* and we call a twisted differential *compatible with* Γ if it moreover satisfies the global k-residue-condition (GRC) from [BCGGM2]. A *multi-scale k-differential* is a twisted differential compatible with Γ up to projectivization of the lower levels, but together with the choice of an equivalence class of prong-matchings. The details are given in Section 3 using the notion of level rotation torus. Leaving them aside, we can now describe the coordinates.

Proposition 1.2. *In a neighborhood $U \subset \Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ of every point in the boundary stratum corresponding to an enhanced level graph Γ with $L+1$ levels there is an orbifold chart given by the perturbed period map*

$$\mathrm{PPer}\colon U \to \mathbb{C}^h \times \mathbb{C}^{L+1} \times \prod_{i=0}^{L} \mathbb{C}^{\dim E^{\mathrm{grc}}_{(-i)}-1},$$

where $E^{\mathrm{grc}}_{(-i)}$ is some eigenspace in homology constrained by the GRC and where the corresponding coordinates are obtained by integrating perturbations of the twisted differential against these homology classes.

In this proposition, the first set of coordinates in $\mathbb{C}^h$ measure the opening of horizontal nodes and the second set in $\mathbb{C}^L$ measures the rescaling of the differentials on each level. Neither of them is a period, in fact they are exponentials, respectively roots, of periods. The statement about integration is intentionally vague, since we are not exactly integrating the (roots of) k-differentials parameterized by U, but its sum with a modification differential, as constructed in Section 3. Moreover, the path of integration is not between the zeros of those differentials but between neighboring points, thus the name 'perturbed'. Technically important is that these perturbations go to zero faster than the rescaling of the k-differential. The map PPer depends on many choices, however they are irrelevant for many local computations.

Boundary divisors. To a first approximation the boundary divisors, i.e. the irreducible components of the boundary $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu) \smallsetminus \Omega^k_+\mathcal{M}_{g,n}(\mu)$, are given by graphs with one level and a single horizontal edge, and by graphs with two levels and no horizontal edge. However, in the setting of k-differentials the level graph does not specify the boundary divisor uniquely. In Section 2 we recall the notion of canonical k-cover, which is unique for k-differentials on smooth curves, but not in the stable case. An example for two different covers that give rise to different components of the boundary is given by [BCGGM2, Figure 2]. In fact, the residue conditions are different in the two cases. Consequently, as second approximation the choice of a cyclic k-cover $\pi\colon \widehat{\Gamma} \to \Gamma$ compatible with the canonical covers of the components (see Section 2) characterizes boundary components.

For a full description of the boundary, and also of its orbifold structure, we need the notion of prong-matchings and the definition of several groups associated with enhanced level graphs. A *prong-matching* at an edge e is a cyclic-order preserving bijection of the κ_e in- resp. outgoing prongs (separatrices) at the two ends of the node corresponding to e, see Section 3. To describe various group actions on prong-matchings, we view $\widehat{\Gamma}$ as a graph with L level *passages*, the first from level 0 to level -1, the second from level -1 to level -2 etc. The unit vector e_i in the *level rotation group* $R_{\widehat{\Gamma}} \cong \mathbb{Z}^L$ acts on the set of prong-matchings by shifting each edge crossing the i-th level passage by one counterclockwise turn. The relevance of this action stems from the level-wise rotation action by $(\mathbb{C}^*)^L$ on the level components of a twisted differential. Of particular importance is the subgroup $\mathrm{Tw}_{\widehat{\Gamma}}$ of $R_{\widehat{\Gamma}}$ that fixes all prongs, the *twist group*.

Proposition 1.3. *There is a bijection between the set of covers $\pi\colon\widehat{\Gamma}\to\Gamma$ of enhanced level graphs Γ of type (g,n,k,μ) and the boundary strata $D_{\widehat{\Gamma}}$ of the compactification $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$. Each $D_{\widehat{\Gamma}}$ is commensurable to the product of the level-wise projectivised moduli space of twisted differentials on $\widehat{\Gamma}$.*

We will not address the subtle question of connectivity of those $D_{\widehat{\Gamma}}$. The details of the construction of a space that admits a finite covering to both $D_{\widehat{\Gamma}}$ and the product level-wise projectivised moduli spaces is given in [CMZ20, Section 4.2]. There, the construction is given for Abelian differentials, but it can verbatim be applied for k-differentials, too.

The metric. We now return to our primary goal. The statement is about flat surfaces of finite area, so we suppose from now on that $m_i > -k$. If $\pi\colon\widehat{X}\to X$ denotes the canonical covering associated with $(X,q)\in\Omega^k\mathcal{M}_{g,n}(\mu)$ such that $\pi^*q=\omega^k$, then the definition

$$h(X,q) \;=\; \operatorname{area}_{\widehat{X}}(\omega) \;=\; \frac{i}{2}\int_{\widehat{X}}\omega\wedge\overline{\omega} \tag{1}$$

provides the tautological bundle $\mathcal{O}(-1)$ on $\mathbb{P}\Omega^k\mathcal{M}_{g,n}(\mu)$ with a hermitian metric. The moduli space $\mathbb{P}\Omega^k\mathcal{M}_{g,n}(\mu)$ has, besides the nice compactification $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ discussed above, a highly singular compactification, the *incidence variety compactification* $\overline{\mathbb{P}\Omega^k\mathcal{M}_{g,n}}(\mu)$ that has been studied in [BCGGM1] and [BCGGM2]. It is the closure of $\mathbb{P}\Omega^k\mathcal{M}_{g,n}(\mu)$ inside the projectivized bundle of k-fold stable differentials twisted by the polar part of μ. This projectivized bundle provides an extension of the tautological bundle $\mathcal{O}(-1)$, whose restriction to the incidence variety compactification we denote by the same symbol.

There is a natural forgetful map $\varphi\colon\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)\to\overline{\mathbb{P}\Omega^k\mathcal{M}_{g,n}}(\mu)$, which is an isomorphism restricted to $\mathbb{P}\Omega^k\mathcal{M}_{g,n}(\mu)$. The pullback of $\mathcal{O}(-1)$ thus provides an extension of the tautological bundle on $\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ that we still denote by the same symbol. It is this bundle whose Chern classes are relevant ([Sau18], [CMSZ19]) for computation of Masur-Veech volumes and Siegel-Veech constants. Our main theorem is:

Theorem 1.4. *The curvature form $\frac{i}{2\pi}[F_h]$ of the metric h is a closed current on $\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ that represents the first Chern class $c_1(\mathcal{O}(-1))$. More generally, the d-th wedge power of the curvature form represents $c_1(\mathcal{O}(-1))^d$ for any $d\geq 1$.*

In an earlier version of the paper we had claimed that the metric h is good in the sense of Mumford. This is not true at boundary points where there are both horizontal and vertical edges, as explained in Section 4. We thank Duc-Manh Nguyen for bringing this to our attention.

More precisely, in the case of only horizontal nodes the metric diverges as we approach the boundary. However in perturbed period coordinates coordinates the local calculation is essentially the calculation of Mumford for the special case of elliptic curves (times the number of horizontal nodes).

In the absence of horizontal nodes, the metric smoothly extends. This fits with the intuition that the area of the lower level surfaces goes to zero. In the presence of both horizontal and vertical edges we estimate directly the growth of the curvature form to justify Theorem 1.4.

This behavior should be contrasted with the one of the flat area metric on the full Hodge bundle $\mathbb{P}\Omega\mathcal{M}_g$ for the principal stratum of abelian differentials. This compactification is smooth and also has a nice (plumbing) coordinate system. However consider a stable curve (X,ω) with two components, joined by two nodes and a differential that is zero on one component, while non-zero on the other. Arbitrarily small neighborhoods contain differentials supported on both components with non-zero residue at the nodes, and thus with infinite volume. As a conclusion, vol: $\Omega\mathcal{M}_g \to \mathbb{R}\cup\{\infty\}$ is not continuous and we thus avoid this space entirely here.

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2. Period coordinates and canonical covers of k-differentials

In this section we summarize well-known results about period coordinates, but also recall the period coordinates along the boundary strata of the incidence variety compactification from [BCGGM2]. We start by recalling properties of the canonical k-cover.

Let X be a Riemann surface and let q be a meromorphic k-differential of type μ. This datum defines (see e.g. [BCGGM2, Section 2.1]) a k-fold cover $\pi\colon \widehat{X}\to X$ such that $\pi^*q=\omega^k$ is the k-power of an abelian differential. This differential is of type

$$\widehat{\mu} := \Big(\underbrace{\widehat{m}_1,\dots,\widehat{m}_1}_{\gcd(k,m_1)},\underbrace{\widehat{m}_2,\dots,\widehat{m}_2}_{\gcd(k,m_2)},\dots,\underbrace{\widehat{m}_n,\dots,\widehat{m}_n}_{\gcd(k,m_n)}\Big)\,,$$

where $\widehat{m}_i := \frac{k+m_i}{\gcd(k,m_i)}-1$. We let $\widehat{g}=g(\widehat{X})$ and $\widehat{n}=\sum_i \gcd(k,m_i)$. Note that $\widehat{X}$ is disconnected, if q is a d-th power of a k/d-differential for some $d>1$.

We fix *once and for all a k-th root of unity ζ*. The Deck group of π contains a unique element τ such that $\tau^*\omega=\zeta\omega$. We fix this automorphism as well. We denote by $\mathbf{p}=(p_1,\dots,p_n)$ the tuple of marked points in X. Similarly, we denote by $\widehat{\mathbf{p}}$ the tuple of preimages of marked points in $\widehat{X}$. The labeling is not canonical, even if we suppose that $\tau(z_i)=z_{i+1}$ within a set of preimages of a fixed point. We thus consider $\widehat{X}$ as partially labelled, two labelings being equivalent if they differ by the renaming the labels within a fiber of π.

For the analogous statements about coverings in the stable case we first need to define twisted k-differentials and further preparation. An *enhanced level graph* for k-differentials is a level graph together with an enhancement map $\kappa: H\to\mathbb{Z}$ on the half-edges, satisfying the following properties:

i) If h and h' are paired to an edge, then $\kappa(h)+\kappa(h')=0$.
ii) At a leg $h\in H\smallsetminus E$ with order m_i, we impose that $\kappa(h)=m_i+k$.
iii) At each vertex $v\in V(\Gamma)$

$$k(2g(v)-2) \;=\; \sum_{h\vdash v}(\kappa(h)-k)\,.$$

This completes the definition of twisted k-differentials and those compatible with a level graph Γ given in the introduction. The global residue condition uses the fact that also a twisted differential (X,q) defines canonical cyclic coverings $\pi\colon\widehat{X}\to X$ such that $\pi^*q=\omega^k$ is the k-power of an abelian differential. Here however the edge identification are in general not uniquely determined by the requirement of a cyclic cover. Most of the local statement in the sequel depend on the specification of an identification that we record as a covering of graphs $\pi\colon \widehat{\Gamma} \to \Gamma$.

We denote by $\mathfrak{W}^k(\widehat{\Gamma})$ the moduli space of twisted k-differential compatible with $\widehat{\Gamma}$, suppressing the dependence on the initial type μ. By the definition of the global residue condition and the main theorem of [BCGGM2], all these differentials are smoothable to some k-differential on a smooth curve. Most relevant for us is the subvariety $\mathfrak{W}^k_{\mathrm{na}}(\widehat{\Gamma})$ parametrizing twisted k-differentials smoothable to a k-differential that is not a k-th power of an abelian differential. Points in $\mathfrak{W}^k_{\mathrm{na}}(\widehat{\Gamma})$ are usually denoted by $(X, \mathbf{q})$, or even more often by $(X, \boldsymbol{\omega})$, where $\mathbf{q} = (q_i)_{i \in L(\Gamma)}$ is a collection of k-differentials on the levels of X, or alternatively $\boldsymbol{\omega} = (\omega_i)_{i \in L(\widehat{\Gamma})}$ is a collection of one-forms on the cover $\widehat{X}$ of X induced by $\widehat{\Gamma} \to \Gamma$.

Next we define the subspaces of homology that we use for period coordinates. We fix some reference smooth surface Σ of genus g with n marked points that we partition as $P \cup Z$ according to the poles of order $\leq k$ among μ and the 'zeros', i.e. with order $> k$. We let $\widehat{\Sigma}$ be a model for the canonical covering surface, which is of genus $\widehat{g}$, and which comes with a map $\pi\colon \widehat{\Sigma} \to \Sigma$. We let $\widehat{P}$ and $\widehat{Z}$ be the preimages of P and Z. They now correspond indeed to the zeros and poles of the type $\widehat{\mu}$.

$$\begin{array}{ccc} \widehat{P}, \widehat{Z} & \widehat{\Sigma} & \widehat{\Lambda}^\circ \supset \widehat{\Lambda} \\ \downarrow \pi & \downarrow \pi & \downarrow \pi \\ P, Z & \Sigma & \Lambda^\circ \supset \Lambda \end{array}$$

If X is a stable curve and π a covering as above, we may find a multicurve $\widehat{\Lambda}$ in $\widehat{\Sigma}$ mapping under π to the multicurve Λ in Σ, such that $\widehat{X}$ and X are obtained by pinching $\widehat{\Sigma}$ and Σ along $\widehat{\Lambda}$ and Λ respectively.

Recall ([BCGGM2, Section 2]) that the moduli space $\Omega^k\mathcal{M}_{g,n}(\mu)$ (and thus also $\Omega^k_+\mathcal{M}_{g,n}(\mu)$) is locally modelled on the ω-periods of the eigenspace

$$E(\widehat{\Sigma} \setminus \widehat{P}, \widehat{Z}) \;=\; H_1(\widehat{\Sigma} \setminus \widehat{P}, \widehat{Z}, \mathbb{C})_{\tau=\zeta}\,.$$

Similarly, we can describe local coordinates for the components of a twisted differential on a stable curve X with enhanced level graph Γ (not yet imposing full compatibility, i.e. the GRC). Let Λ° be an open thickening of Λ. We let $\Lambda^\pm$ be the upper and lower boundaries of Λ°. The level structure on Γ organizes $\Sigma \setminus \Lambda$ into levels $\Sigma_{(i)}$ and we denote the adjacent poles, zeros and boundaries $\Lambda^\pm$ with the subscript (i). All the notation apply with a hat to the corresponding objects on the k-cover. The level-i component of the twisted differential is thus modelled on

$$E_{(i)} \;=\; H_1(\widehat{\Sigma}_{(i)} \setminus \{\widehat{P}_{(i)} \cup \widehat{\Lambda}^\circ_{(i)}\}, \widehat{\Lambda}^+_{(i)} \cup \widehat{Z}_{(i)}, \mathbb{C})_{\tau=\zeta}\,. \tag{2}$$

We can now restate the main dimension estimate in the proof of [BCGGM2, Theorem 6.2].

Proposition 2.1. *Twisted differentials compatible with an enhanced level graph are locally modelled on the $\omega_{(-i)}$-periods of $\prod_{i=0}^L E^{\mathrm{grc}}_{(-i)}$, where $E^{\mathrm{grc}}_{(i)} \subseteq E_{(i)}$ is the subspace at level $i \in L(\Gamma)$ cut out by the global residue condition. The dimensions of these subspaces*

$$\sum_{i=0}^{L} \dim_{\mathbb{C}} E^{\mathrm{grc}}_{(-i)} \;=\; \dim_{\mathbb{C}} \Omega^k\mathcal{M}_{g,n} \,-\, h$$

add up to the total dimension of the moduli space of k-differentials of type μ minus the number of horizontal edges h.

3. Construction of $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$

In this section we recall the main technical tools from [BCGGM3], construct the compactification and eventually prove Theorem 1.1. The definitions in Section 3.1–Section 3.4 are direct adaptation of the abelian case by working on the canonical k-covers. Avoiding the discussion of Teichmüller spaces means omission of that aspect but also a simplification of notations. In the remaining sections we have to ensure at some places that constructions can be performed τ-equivariantly.

3.1. **Degeneration, undegeneration.** We describe here two types of maps between level graphs Γ that encode the degeneration of curves, together with the compatible maps between the coverings graphs $\widehat{\Gamma}$ that form part of the degeneration datum. In fact, it is easier to first describe the inverse process of undegeneration that encodes all the k-differentials in a neighborhood of a given degenerate k-differentials.

Let $\pi\colon \widehat{\Gamma} \to \Gamma$ by a cyclic k-covering of enhanced level graphs with $L+1$ normalized levels. For any subset $I \subset \{-1, \ldots, -L\}$, to be memorized as the *the level passages that remain*, we define the *vertical undegeneration* δ_I as the following contraction of certain vertical edges. An edge e is contracted by δ_I if and only if the levels $e^-, e^-+1, \ldots, e^+-1$ all belong the the complement of I. Here the symbol e^- denotes both the point in $\widehat{X}$ at the bottom end of the edge e and its level. The meaning should be clear from the context. Similarly, e^+ refers to the top end. This edge contraction is performed simultaneously on the domain and range of π and induces a cyclic k-covering $\delta_I(\widehat{\Gamma}) \to \delta_I(\Gamma)$ that we abbreviate as $\delta_I(\pi)$. We write $\delta_I(j)$ for the image of the j-th level under δ_I.

Moreover, we define for any subset $E_0 \subset E^{\rm hor}$ of the horizontal edges of Γ the *horizontal undegeneration* δ_{E_0} to be the edge contraction that contracts precisely the edges in E_0 in Γ. Contracting simultaneously also on the π-preimages of E_0 in $\widehat{\Gamma}$, we obtain a new cyclic k-covering $\delta_{E_0}(\pi) : \delta_{E_0}(\widehat{\Gamma}) \to \delta_{E_0}(\Gamma)$.

A general undegeneration is a pair $\delta = (\delta_I, \delta_{E_0})$, defined by composing a horizontal and a vertical undegeneration in either order. A *degeneration* is the inverse of an undegeneration. We write $\widehat{\Gamma}' \rightsquigarrow \widehat{\Gamma}$ for a general degeneration of level graphs and $\delta^{\rm ver}$ and $\delta^{\rm hor}$ for the two constituents of an undegeneration δ.

3.2. **Prong-matchings as extra structure on twisted differentials.** We start with the definition of prong-matchings and the welded surfaces constructed from these. Given a differential ω on X that has been put in standard from $z^\kappa dz/z$ if $\kappa \geq 0$ or $(z^\kappa + r)dz/z$ if $\kappa \leq -2$, the *prongs* are the $|\kappa|$ tangent vectors $e^{2\pi i j/|\kappa|}\frac{\partial}{\partial z}$ for $\kappa > 0$ and $-e^{2\pi i j/|\kappa|}\frac{\partial}{\partial z}$ for $j = 0, \ldots, |\kappa - 1|$. At simple poles, prongs are not defined.

We now get back to a twisted differential $(\widehat{X}, \boldsymbol{\omega}, \widehat{\Gamma})$. Define a *local prong-matching* σ_e at the vertical edge e of $\widehat{\Gamma}$ to be a cyclic order-reversing bijection between the prongs at the upper and lower end of e. A *global prong-matching* is a collection $\sigma = (\sigma_e)_{e \in E(\widehat{\Gamma})}$ of local prong-matchings that is equivariant with respect to the action of τ permuting the edges and multiplying the local coordinates z by ζ.

A global prong-matching σ on $\widehat{X}$ gives an *almost-smooth surface* $\widehat{X}_\sigma$, i.e. a smooth surface except for nodes corresponding to the horizontal nodes of $\widehat{X}$, constructed by the following procedure of *welding*. Take the partial normalization of $\widehat{X}$ separating branches at vertical nodes and perform the real oriented blowup of each pair of preimages. Then identify the boundary circles isometrically so as to identify boundary points that are paired by the prong-matching. More details of the construction can be found in Section 4 of [BCGGM3], see also [ACG11]. (We only use subscripts σ to denote weldings here and suppress the overline used in loc. cit. to avoid double decorations.) Horizontal nodes remain untouched in the welding procedure.

On almost-smooth surfaces any *good arc* γ, i.e. any arc transversal to the seams created by welding, has a well-defined *turning number*, that we denote by $\rho(\gamma)$.

Adding the information of prong-matching to points in $\mathfrak{W}^k(\widehat{\Gamma})$ will give us a finite covering space. We will next construct this covering in detail for the subset $\mathfrak{W}^k_{\mathrm{na}}(\widehat{\Gamma})$ of twisted differentials admitting a non-abelian smoothing, since we will be exclusively concerned with that space.

We define the set $\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})$ to be tuples $(X, \boldsymbol{\omega}, \sigma)$ consisting of a point $(X, \boldsymbol{\omega}) \in \mathfrak{W}^k_{\mathrm{na}}(\widehat{\Gamma})$ together with a prong-matching σ. There is an obvious notion of parallel transport of prong-matchings that allows to lift inclusions of contractible open sets $U \to \mathfrak{W}^k_{\mathrm{na}}(\widehat{\Gamma})$ uniquely to maps $U \to \mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})$. Requiring that these lifts are holomorphic local homeomorphism provides $\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})$ with a complex structure so that $\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma}) \to \mathfrak{W}^k_{\mathrm{na}}(\widehat{\Gamma})$ is a covering map.

3.3. **The level rotation torus.** Our compactification combines the geometry of moduli spaces of k-differentials of lower complexity and aspects of a toroidal compactification. The torus action for the latter is given by the level rotation torus that we now define.

In the introduction we defined the *twist group* to be the (full rank) subgroup $\mathrm{Tw}_{\widehat{\Gamma}}$ of the level rotation group $R_{\widehat{\Gamma}} \cong \mathbb{Z}^L$ that fixes all prongs. The *(reduced) level rotation torus* $\mathrm{T}_{\widehat{\Gamma}}$ is the quotient

$$\mathrm{T}_{\widehat{\Gamma}} \;=\; \mathbb{C}^L / \mathrm{Tw}_{\widehat{\Gamma}}\,.$$

(Here reduced refers to the fact that $\mathrm{T}_{\widehat{\Gamma}}$ does not rotate the top level. We will introduce this action separately for projectivization and usually drop the adjective 'reduced'.) The level rotation torus can also be characterized ([BCGGM3, Proposition 5.4]) as the connected component of the identity of the subtorus

$$\Big\{ ((r_i, \rho_e))_{i,e} \in (\mathbb{C}^*)^L \times (\mathbb{C}^*)^{E(\widehat{\Gamma})} \,|\, r_{e^-} \dots r_{e^+-1} = \rho_e^{\kappa_e} \text{ for all } e \in E(\widehat{\Gamma}) \Big\}. \tag{3}$$

This characterization makes the reason for introducing $\mathrm{T}_{\widehat{\Gamma}}$ apparent, as there is a natural action of the level rotation torus on $\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})$ given by

$$\begin{array}{rcl} \mathrm{T}_{\widehat{\Gamma}} \times \mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma}) & \to & \mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma}) \\ (r_i, \rho_e) * (\widehat{X}, (\omega_{(i)}), (\sigma_e)) & = & \Big(\widehat{X}, (r_i \dots r_{-1} \omega_{(i)}), (\rho_e * \sigma_e) \Big) \end{array} \tag{4}$$

where $\rho_e * \sigma_e$ is the prong-matching σ_e post-composed with the rotation by $\arg(\rho_e)$ (if the full Dehn twist around e corresponds to angle 2π, equivalently by the rotation by $\kappa \arg(\rho_e)$ for the angle in the flat metric). We alert the reader that this action uses the 'triangular' basis, where the i-th component of $\mathrm{T}_{\widehat{\Gamma}}$ rotates the i-th level and all level below it by the amount r_i.

The toric variety associated with the level rotation torus will not be smooth in general. To obtain orbifold charts we define first the *simple twist group* $\mathrm{Tw}^s_{\widehat{\Gamma}} \subseteq \mathrm{Tw}_{\widehat{\Gamma}}$ as the twist group elements generated by rotations of one level at a time, i.e.

$$\mathrm{Tw}^s_{\widehat{\Gamma}} \;=\; \oplus_{i=1}^L \mathrm{Tw}_{\delta_{-i}(\widehat{\Gamma})}\,.$$

We can now define the *(reduced) simple level rotation torus* as

$$\mathrm{T}^s_{\widehat{\Gamma}} \;=\; \mathbb{C}^L/\mathrm{Tw}^s_{\widehat{\Gamma}}\,. \tag{5}$$

In order to describe the action of these tori we will need the integers

$$\ell_i \;=\; \mathrm{lcm}_{e\in E(\delta_i(\widehat{\Gamma}))} k_e\,, \quad \text{and} \quad m_{e,i} \;=\; \ell_i/\kappa_e \tag{6}$$

for $i = -1,\ldots,-L$ and $e \in E(\widehat{\Gamma})$. Now Proposition 5.4 in loc. cit. moreover shows that there is an identification $\mathrm{T}^s_{\widehat{\Gamma}} \cong (\mathbb{C}^*)^L$ such that the projection $\mathrm{T}^s_{\widehat{\Gamma}} \to \mathrm{T}_{\widehat{\Gamma}}$ is given in coordinates by

$$(s_i) \;\mapsto\; (r_i, \rho_e) \;=\; \Big(s_i^{\ell_i}\,,\, \prod_{i=e^-}^{e^+-1} s_i^{\ell_i/\kappa_e}\Big) \tag{7}$$

The composition of this parametrizations (7) of $\mathrm{T}_{\widehat{\Gamma}}$ by $\mathrm{T}^s_{\widehat{\Gamma}}$ with the action (4) gives an action of $\mathbf{s} = (s_i) \in \mathrm{T}^s_{\widehat{\Gamma}}$ on welded surfaces and we denote the image of $\widehat{X}_\sigma$ under the action of $\mathbf{s}$ by $\widehat{X}_{\mathbf{s}\cdot\sigma}$.

3.4. **The compactification as topological space.** We start with the definition of $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ as a set. For each k-cyclic covering $\pi : \widehat{\Gamma} \to \Gamma$ we define the *boundary stratum* $\Omega^k\mathcal{B}_{\widehat{\Gamma}} = \mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})/\mathrm{T}_{\widehat{\Gamma}}$ and we define the set

$$\Xi^k\overline{\mathcal{M}}_{g,n}(\mu) \;=\; \coprod_{\pi:\widehat{\Gamma}\to\Gamma} \Omega^k\mathcal{B}_{\widehat{\Gamma}}\,. \tag{8}$$

This union includes $\Omega^k_+\mathcal{M}_{g,n}$ for π being the trivial covering of a point to a point. Points of $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ are called *multi-scale k-differentials*, i.e. the preceding definition completes the specification of the equivalence relation stated in the introduction. Points of $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ are thus given by a tuple $(X, \mathbf{p}, \widehat{\Gamma}, \boldsymbol{\omega}, \sigma)$ where $\boldsymbol{\omega} = (\omega_{(-i)})_{i=0}^L$ is a tuple indexed according to the levels. We often write just $(X, \boldsymbol{\omega})$ or $(X, \boldsymbol{\omega}, \widehat{\Gamma})$. The equivalence classes are given by the orbits of the action (4) on $(\boldsymbol{\omega}, \sigma)$.

We now provide this space with a topology by exhibiting all converging sequences. The basic idea is the conformal topology on $\overline{\mathcal{M}}_g$ where sequences converge if there is an exhaustion of the complement of nodes and punctures and conformal maps of the exhaustion to neighboring surfaces, see (b) below. For multiscale differentials we require moreover the convergences of the differentials as in (c) after a rescaling, where the magnitude of rescaling is compatible with the level structure, see (a) and (c). Since the conformal topology only requires the comparison maps to be diffeomorphisms near the nodes, which can twist arbitrarily, we need to add (d) to avoid constructing a non-Hausdorff space. In the sequel we write $\widehat{X}_{\sigma_n}$ for $(\widehat{X}_n)_{\sigma_n}$ in a sequence of welded surfaces.

We say that a sequence $(\widehat{X}_n, \boldsymbol{\omega}_n, \widehat{\Gamma}_n)$ converges to $(\widehat{X}, \boldsymbol{\omega}, \widehat{\Gamma})$, if there exist representatives of all the equivalence classes (that we denote by the same letters), a sequence $\varepsilon_n \to 0$ and a sequence $\mathbf{s}_n = (s_{n,i})_{i=-L}^{-1} \in (\mathbb{C}^*)^L$ of tuples such that:

(a) For sufficiently large n there is an undegeneration $\delta_n = (\delta_n^{\mathrm{ver}}, \delta_n^{\mathrm{hor}})$ with $\delta_n^{\mathrm{ver}}(\widehat{\Gamma}_n) = \Gamma$.
(b) For sufficiently large n there is an almost-diffeomorphism $g_n\colon \widehat{X}_{\mathbf{s}_n \cdot \sigma} \to \widehat{X}_{\sigma_n}$ that is conformal on the ϵ_n-thick part of $(\widehat{X}, \widehat{\mathbf{p}})$ and that respects the marked points, up to relabeling in π-fibers.
(c) The restriction of $\prod_{j=i}^{-1} s_{n,j}^{\ell_j} \cdot g_n^*(\omega_n)$ to the ϵ_n-thick part of the level i subsurface of $(\widehat{X}, \widehat{\mathbf{p}})$ converges uniformly to $\omega_{(i)}$.
(d) For any $i, j \in L(\widehat{\Gamma})$ with $i > j$, and any subsequence along which $\delta_n^{\mathrm{ver}}(i) = \delta_n^{\mathrm{ver}}(j)$, we have
$$\lim_{n\to\infty} \prod_{k=j}^{i-1} |s_{n,k}|^{-\ell_k} = 0\,.$$
(e) The almost-diffeomorphism g_n are nearly turning-number preserving, i.e. for every good arc γ in $\widehat{X}_\sigma$, the difference $\rho(g_n \circ F_{\mathbf{s}_n} \circ \gamma) - \rho(F_{\mathbf{s}_n} \circ \gamma)$ of turning numbers converges to zero, where $F_{\mathbf{s}_n}$ is the fractional Dehn twist around the edge e by the angle $\prod_{j=1}^{i} s_{n,j}^{\ell_j/\kappa_e}$.

This topology is exactly the topology of the compactification of the moduli spaces $\Omega\mathcal{M}_g(\widehat{\mu})$ in [BCGGM3] restricted to the subspace of k-cyclic covers. Note that the inclusion of $\widehat{\Gamma}$ into the datum of a multi-scale k-differential implies that even boundary points have canonically determined k-covers. We may thus view
$$\Xi^k\overline{\mathcal{M}}_{g,n}(\mu) \subset \Xi\overline{\mathcal{M}}_{\widehat{g},\{\widehat{n}\}}(\widehat{\mu}) \tag{9}$$
with the subspace topology, where $\{\widehat{n}\}$ indicates that we have taken partial labelings, the quotient by the action of the group permuting the labels within the group of $\gcd(k, m_j)$ labels points of type $\widehat{m_j}$ for all $j = 1, \ldots, n$.

Proposition 3.1. *The moduli space $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ is a Hausdorff topological space and its projectivization $\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ is a compact Hausdorff space.*

Proof. This follows from the definition of the subspace topology, the fact that being a k-cover is a closed condition and [BCGGM2, Theorem 9.4 and Proposition 14.2]. Alternatively, those proofs can be adapted directly to the current situation without Teichmüller markings. □

3.5. **Model differentials and modification differentials.** In order to provide $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ with a complex structure we use a local model space that automatically has a complex structure (as a finite cover of a product of spaces of non-zero k-differentials). The degeneration of differentials on lower components is emulated in the model space by vanishing of auxiliary parameters t_i.

The action (4) of the simple level rotation torus $T^s_{\widehat{\Gamma}}$ makes $\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})$ into a principal $(\mathbb{C}^*)^L$-bundle over the 'simple' version of the boundary stratum $\Omega^k\mathcal{B}^s_{\widehat{\Gamma}} = \mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})/T^s_{\widehat{\Gamma}}$ and we define the *(compactified) simple model domain* $\overline{\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})^s}$ to be the associated $\mathbb{C}^L$-bundle over $\Omega^k\mathcal{B}^s_{\widehat{\Gamma}}$. The construction directly implies:

Proposition 3.2. *The compactified simple model domain $\overline{\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})^s}$ is smooth with normal crossing boundary divisor. If t_i is a coordinate on $\mathbb{C}^L$, then the boundary divisor $D_i = \{t_i = 0\}$ corresponds to model differentials compatible with $\delta_i(\widehat{\Gamma})$ and its degenerations.*

The space $\mathfrak{W}^k_{\mathrm{na}}(\widehat{\Gamma})$, being just a GRC-subspace in a product of moduli spaces, obviously comes with a universal family of curves and k-differentials that we can pull back to $\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})$. Since the level rotation torus only acts on differentials and prong-matchings, not on the curve, the universal curve descends to a family of curves $f\colon \widehat{\mathcal{X}} \to \overline{\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})^s}$. Over small enough open sets $W \subset \overline{\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})^s}$ (even at the boundary!) we can fix a scale of the $\mathrm{T}^s_{\widehat{\Gamma}}$-orbits and work with a collection $\boldsymbol{\eta} = (\eta_{(i)})_{i=0}^L$ of families of differentials, not identically zero on any component of any fiber. From now on with stick to the convention that $\boldsymbol{\eta}$ denotes (families of) model differentials (that come with degeneration parameters t_i) whereas $\boldsymbol{\omega}$ denotes (families of) multi-scale differentials (that may become zero on components of special fibers). We alert the reader of the inevitable notation problem that for trivial families (i.e. just a single surface $\widehat{X}$) multi-scale-k-differentials are equivalence classes of prong-matched twisted differentials, thus both denoted by $\boldsymbol{\omega}$, and model-differentials are the same objects, albeit denoted by $\boldsymbol{\eta}$. It is only in degenerating families that the difference becomes apparent.

Note that the boundary of the compactified simple model domain comes with a natural *stratification* given by the subset of $\{-1, \dots, -L\}$ of the t_i that are zero.

Modification differentials will be used for plumbing and also for perturbed period coordinates on charts of $\overline{\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})^s}$. In the sequel we check that the setup of [BCGGM3, Section 11] works in the k-equivariant setup. We define

$$\mathbf{t} * \boldsymbol{\eta} \;=\; \left(t_{\lceil i \rceil} \cdot \eta_{(i)}\right)_{i \in L(\widehat{\Gamma})} \;=\; \left(t_{-1}^{\ell_{-1}} \dots t_i^{\ell_i} \cdot \eta_{(i)}\right)_{i \in L(\widehat{\Gamma})}, \tag{10}$$

for $\mathbf{t} = (t_{-1}, \dots, t_{-L}) \in (\mathbb{C}^*)^L$.

Definition 3.3. *A* equivariant family of modifying differentials *over W equipped with the universal differential $\mathbf{t} * \boldsymbol{\eta}$ is a family of meromorphic differentials $\boldsymbol{\xi} = (\xi_{(i)})_{i=0}^{-L}$ on $f\colon \widehat{\mathcal{X}} \to W$, such that*

(i) *the equivariance $\tau^* \boldsymbol{\xi} = \zeta \cdot \boldsymbol{\xi}$ holds,*
(ii) *the differentials $\xi_{(i)}$ are holomorphic, except for possible simple poles along both horizontal and vertical nodal sections, and except for marked poles,*
(iii) *the component $\xi_{(-L)}$ vanishes identically and moreover $\xi_{(i)}$ is divisible by $t_{\lceil i-1 \rceil}$ for each $i = -1, \dots, -L+1$, and*
(iv) *the sum $\mathbf{t} * \boldsymbol{\eta} + \boldsymbol{\xi}$ has opposite residues at every node.*

Proposition 3.4. *The universal family $f\colon \widehat{\mathcal{X}} \to W$ equipped with the universal differential $\mathbf{t} * \boldsymbol{\eta}$ admits an equivariant family of modifying differentials.*

Proof. The proof of [BCGGM3, Proposition 11.3] works in the situation where the edges of $\widehat{\Gamma}$ are images of the pinched multicurve Λ via a family of markings $\Sigma \to \widehat{X}$ by a reference surface Σ. Choosing W contractible, we may assume that we have such a marking here as well.

The proof in loc. cit. starts by taking the subspace $V = \langle \lambda \in \Lambda \rangle_{\mathbb{Q}}$ and the subspace V_P spanned by the loops around the marked poles inside $H_1(\widehat{X} \setminus \widehat{P}, \mathbb{Q})$.

We define $V_N = V + V_P$. The proof proceeds by searching for a complementary subspace V_C (i.e. with $V_N \cap V_C = 0$) such that the projection $p(V')$ to $H_1(\widehat{X}, \mathbb{Q})$ is a Lagrangian subspace, where $V' = V_N + V_C$. The proof then constructs $\boldsymbol{\xi} = (\xi_{(i)})$ for each $w \in W$ from assignments $\rho_i : V_i \to \mathbb{C}$ determined by the periods of the fiber $\boldsymbol{\eta}_w$ on subspaces V_i of $V + V_P$ generated by multicurves associated to edges whose lower level is below i. Relevant here is that those $\boldsymbol{\xi}$ satisfy all properties of Definition 3.3 except possibly the equivariance in (i). Moreover, $\boldsymbol{\xi}$ depends uniquely on an extension ρ'_i of ρ_i, that we may chose to be zero on V_C.

If we can find a subspace V_C which is τ-invariant, then the extended residue assignment ρ'_i is τ-equivariant (with τ acting by multiplication by ζ on the range) and thus $\boldsymbol{\xi}$ satisfies (i). To find such a V_C, we enlarge V_C and thus $V' = V_C + V_N$ step by step, staying τ-invariant at each step, until $p(V')$ is a Lagrangian subspace. If at some step V' is τ-invariant, but $p(V')$ is strictly contained in a Lagrangian subspace, we may find an element γ that pairs trivially with $p(V')$. But then $\tau^i(\gamma)$ also pairs trivially with $p(V')$ for all i and we add to V_C the span of all $\tau^i(\gamma)$. $\square$

3.6. **The perturbed period map.** Periods give local coordinates on $\mathfrak{W}^k_{\mathrm{na}}(\widehat{\Gamma})$ and thus on $\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})$. Together with the tuple of degeneration parameters $\mathbf{t}$ and deprived of one coordinate per level to fix the scale of projectivization they give local coordinates of $\overline{\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})^s}$. We introduce some perturbation of these coordinates here and show that this still gives local coordinates. The reason for this procedure is that the perturbed period coordinates can still be used after plumbing, see Section 3.7. Together with horizontal plumbing parameters it will provide coordinates on an orbifold chart of $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$. Except for the use of appropriate eigenspaces this is exactly [BCGGM3, Section 11].

Near the marked point e^+ corresponding to the upper end (say on level $i = i(e^+)$) of each of the vertical nodes, choose an auxiliary section $s_e^+ : W \to \widehat{\mathcal{X}}$ such that

$$\int_{e^+}^{s_e^+(w)} \boldsymbol{\eta}_{(i)} \;=\; \text{const}\,, \tag{11}$$

where the constant is sufficiently small, depending on W, and constrained by the plumbing construction later. Near each zero marked z_j of $\boldsymbol{\eta}$ (say on level $i = i(z_j)$) choose an auxiliary section $s_j : W \to \widehat{\mathcal{X}}$ that coincides with the barycenter of the zeros of $\eta_{(i)} + t_{\lceil i \rceil}^{-1} \cdot \xi_{(i)}$ that result from the deformation of z_j. We let γ_{ij} for $i = 0, \ldots, -L$ and $j = 1, \ldots, \dim E^{\mathrm{grc}}_{(i)}$ be a basis of the subspaces $E^{\mathrm{grc}}_{(i)}$ of homology. Since the contribution of each level to the twisted differentials compatible with a level graph is positive-dimensional (by the rescaling of the differential), the definition of periods coordinates along the boundary in Proposition 2.1 implies that for each i there exists some j such that $\int_{\gamma_{i,j}} \eta_{(i)} \neq 0$. We use this to fix the scale of the projectivization and assume that the periods for $j = 1$ are normalized on each level, i.e. $\int_{\gamma_{i,1}} \eta_{(i)} = 1$.

The i-th level component of the perturbed period map is now given by

$$\mathrm{PPer}_i\colon \begin{cases} \quad W & \to \quad \mathbb{C}^{\dim E^{\mathrm{grc}}_{(i)} - 1 + \delta_{i,0}}\,, \\ [(\widehat{X}, \boldsymbol{\eta}, \mathbf{t})] & \mapsto \quad \left(\int_{\gamma_{i,j}} \eta_{(i)} + t_{\lceil i \rceil}^{-1} \cdot \xi_{(i)} \right)_{j=2-\delta_{i,0}}^{\dim E^{\mathrm{grc}}_{(i)}}\,, \end{cases} \tag{12}$$

where $\delta_{i,0}$ is Kronecker's delta and where the integrals are to be interpreted starting and ending at the nearby points determined by the sections s_e^+ and s_j rather than the true zeros of $\boldsymbol{\eta}$. The reason for this technical step is that those nearby points are still present after the surfaces has been plumbed ('Step 2' below).

Proposition 3.5. *The perturbed period map*

$$\mathrm{PPer}^{\mathrm{MD}}\colon W \to \mathbb{C}^L \times \prod_{i=0}^{-L} \mathbb{C}^{\dim E^{\mathrm{grc}}_{(i)} - 1 + \delta_{i,0}}, \quad [(\widehat{X}, \boldsymbol{\eta}, \mathbf{t})] \mapsto \Big(\mathbf{t}\,;\, \coprod_{i=0}^{-L} \mathrm{PPer}_i(\widehat{X}, \boldsymbol{\eta}, \mathbf{t})\Big)$$

is open and locally injective on a neighborhood of the most degenerate boundary stratum $W_\Lambda = \cap_{i=1}^L D_i$ in the compactified model domain $\overline{\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})^s}$.

We will write $(\mathbf{t}, \mathbf{w}) = \mathrm{PPer}^{\mathrm{MD}}(\widehat{X}, \boldsymbol{\eta}, \mathbf{t})$.

Proof. As in [BCGGM3, Proposition 11.6] it suffices to show that the derivative is surjective, by dimension comparison. For the tangent directions to the boundary this follows from Proposition 2.1 (and the fact that we have projectivized the lower levels). For the transverse direction this follows since the t_i are the local coordinates of the $\mathbb{C}^L$-bundles used to construct the compactifications. □

The reader should keep in mind, that in the model domain with its equisingular family of curves horizontal nodes are untouched. They enter in Proposition 1.2 only after plumbing horizontal nodes, see below.

3.7. **The complex structure and the proof of Theorem 1.1.** The outline of the proof of Theorem 1.1 consists of the following steps.

1) Construct locally covers $U^s \to U$ for small open sets $U \subset \Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ that will be used as orbifold charts.
2) Perform a plumbing construction on the pullback of the universal family $f\colon \widehat{\mathcal{X}} \to \overline{\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})^s}$ to small open sets W and via the second projection to $W \times \Delta^h_\varepsilon$ obtain a family $\mathcal{Y} \to W \times \Delta^h_\varepsilon$ together with a family of differentials.
3) Use the moduli properties of the strata of $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ to obtain moduli maps $\Omega\mathrm{Pl} : W \times \Delta^h_\varepsilon \to U^s$ for appropriately chosen target set U, defined stratum by stratum.
4) Show that $\Omega\mathrm{Pl}$ is a homeomorphism near a central point $P \times (0, \dots, 0) \in W \times \Delta^h_\varepsilon$ and thus provide charts there.

The charts constructed in this way depend on many choices, in the construction of the modification differential and the parameters for plumbing. However, the induced complex structures fit together and that's all we need since $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ already exists as a topological space. We provide the details for Step 1) and Step 2), since there the τ-equivariance needs to be respected and since we need this in the next section. The technical Step 3) and Step 4) proceed exactly as in [BCGGM3].

Step 1. In order to provide $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ with a complex structure we consider the neighborhood U of a point $(X, \boldsymbol{\omega}, \widehat{\Gamma})$ that we may assume to be at the boundary, say for level graph $\widehat{\Gamma}$. (The following description assumes that $(X, \boldsymbol{\omega})$ has no automorphisms. In general we should start from an orbifold chart, and add the extra orbifold structure described below.) The compactified simple model domain is a $K = \mathrm{Tw}_{\widehat{\Gamma}}/\mathrm{Tw}^s_{\widehat{\Gamma}}$-cover of the (in general) singular space that we would get by compactifying the $\mathrm{T}_{\widehat{\Gamma}}$-quotient of $\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})$. Consequently, we have to pass locally near

$(X, \boldsymbol{\omega}, \widehat{\Gamma})$ to a K-cover of U. We define this cover U^s as follows. Define the auxiliary *simple boundary stratum* to be $\Omega^k\mathcal{B}^s_{\widehat{\Gamma}} = \mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})/\mathrm{T}^s_{\widehat{\Gamma}}$. As a set

$$U^s = \Big\{(X', \boldsymbol{\omega}', \widehat{\Gamma}') \in \bigcup_{\widehat{\Gamma}' \rightsquigarrow \widehat{\Gamma}} \Omega^k\mathcal{B}^s_{\widehat{\Gamma}'} \,:\, \varphi((X', \boldsymbol{\omega}', \widehat{\Gamma}')) \in U\Big\}\,,$$

where φ is induced by the quotient maps $\Omega^k\mathcal{B}^s_{\widehat{\Gamma}'} \to \Omega^k\mathcal{B}_{\widehat{\Gamma}'}$. We provide U^s with a topology where convergence is formally given exactly by the same conditions as for $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ in Section 3.4, but where now the 'existence of representatives of the equivalence classes' is up to the torus $\mathrm{T}^s_{\widehat{\Gamma}}$ rather than the quotient torus $\mathrm{T}_{\widehat{\Gamma}}$.

Step 2. To start the plumbing construction we first define the *plumbing fixture* for each vertical edge $e \in E(\widehat{\Gamma})$ to be the degenerating family of annuli

$$\mathbb{V}_e \;=\; \Big\{(\mathbf{w}, \mathbf{t}, u, v) \in W \times \Delta^2_\delta : uv = \prod_{i=e^-}^{e^+-1} t_i^{m_{e,i}}\Big\}, \tag{13}$$

that only depends on the $\mathbf{t}$-part of the perturbed period coordinates $(\mathbf{t}, \mathbf{w})$ of W. We equip $\mathbb{V}_e$ with the family of differentials

$$\Omega_e \;=\; \left(t_{\lceil e^+\rceil} \cdot u^{\kappa_e - 1} - \frac{r'_e}{u}\right) du \;=\; \left(-t_{\lceil e^-\rceil} \cdot v^{-\kappa_e - 1} + \frac{r'_e}{v}\right) dv\,, \tag{14}$$

where we recall that $t_{\lceil i\rceil} \;=\; t_i^{\ell_i} \ldots t_1^{\ell_1}$ and where $r'_e = r'_e(\mathbf{w}, \mathbf{t})$ are the residues of the universal family over model domain. Inside the plumbing fixture we define the gluing annuli $\mathcal{A}^{\pm}_e$ by $\delta/R < |u| < \delta$ and $\delta/R < |v| < \delta$ respectively. The sizes δ, R and the size of the neighborhood W will be determined in terms of the geometry of the universal family, to ensure for example that plumbing annuli are not overlapping.

Suppose we only have vertical nodes. The plumbing construction proceeds bottom up. Near each of the nodes of bottom level we put the family of differentials $\eta_{(-L)}$ in standard form $(v^{-\kappa_e-1} + \frac{r_e}{v})dv$ so that after rescaling with $t_{\lceil e^-\rceil}$ it can be glued to Ω_e for $r'_e = t_{\lceil e^-\rceil} r_e$. That such a normal form exists in families is the content of [BCGGM3, Theorem 3.3]. The functions r'_e determine the modification differential $\xi_{(-L+2)}$ as the proof of Proposition 3.4 shows, see [BCGGM3, Corollary 11.4]. We will thus put $t_{\lceil e^+\rceil}\eta_{(-L)} + \xi_{(-L+1)}$ in standard form near e^+ using the normal form on the deformation of an annulus ([BCGGM1, Theorem 4.5] or [BCGGM3, Theorem 12.2]) and this glues with the form (14) on the upper end of the annulus. Iterating the procedure allows to plumb the collection of families of one-forms

$$\mathbf{t} * \boldsymbol{\eta} + \boldsymbol{\xi} \;=\; \big(t_{\lceil i\rceil} \cdot \eta_{(i)} + \xi_{(i)}\big)_{i=0}^{-L} \tag{15}$$

on the equisingular family of curves $\mathcal{X} \to W$ to a family of one-forms ω on a degenerating family of curves $\mathcal{Y} \to W$.

In the preceding construction we have neglected so far that the choice of the normal form is unique only up multiplication by a κ_e-th root of unity. The prong-matching that is part of the datum of the universal family over the model domain determines this choice. Many more details, using reference sections to make the construction rigorous, are given in [BCGGM3, Section 12].

The whole construction can obviously performed τ-equivariantly, since the modification differential is τ-equivariant and since the sizes of the neighborhoods and plumbing annuli are determined by the rates of degeneration of $\mathbf{t} * \boldsymbol{\eta} + \boldsymbol{\xi}$, i.e. by τ-equivariant data.

Finally, we investigate horizontal nodes of $\widehat{\Gamma}$, that come in τ-orbits of length k and that we thus label as $e_1^{(a)}, \ldots, e_h^{(a)}$ for $0 \le a < k$. We parameterize the plumbing by additional plumbing parameters $\mathbf{x} = (x_1, \ldots, x_h) \in \Delta_\epsilon^h$ and define the *(horizontal) plumbing fixture* to be

$$\mathbb{W}_j \;=\; \left\{(\mathbf{w}, \mathbf{t}, \mathbf{x}, u, v) \in W \times \Delta_\epsilon^h \times \Delta_\delta^2 : uv = x_j \right\}, \tag{16}$$

independently of the upper label a of $e_j^{(a)}$, equipped with the family of holomorphic one-forms

$$\Omega_j \;=\; -r'_{e_j}(\mathbf{w}, \mathbf{t}) du/u \;=\; r'_{e_j}(\mathbf{w}, \mathbf{t}) dv/v\,, \tag{17}$$

where $\pm r'_{e_j}(\mathbf{w}, \mathbf{t})$ is the residue of $\mathbf{t} * \boldsymbol{\eta} + \boldsymbol{\xi}$ at the j-th horizontal node. Here the gluing happens along annuli $\mathcal{B}_j^{\pm}$ by $\delta/R < |u| < \delta$ and $\delta/R < |v| < \delta$.

Step 3. The existence of moduli maps on each stratum of the simple model domain to $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ is immediate from the construction of $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ as union of strata $\Omega^k\mathcal{B}_{\widehat{\Gamma}} = \mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})/\mathrm{T}_{\widehat{\Gamma}}$ and the property of $\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})$ as moduli space of k-differentials. We let U be the range of the union of these maps. The map factors through U^s since both this space and the simple model domain are defined as $\mathrm{T}^s_{\widehat{\Gamma}}$-equivalence classes. [BCGGM3, Section 12.5] provides more details.

Step 4. To show that the resulting map $\Omega\mathrm{Pl}\colon W \times \Delta^h \to U^s$ is continuous we have to invoke the definition of the topology on U^s to show that the images of a converging sequence converges. This entails exhibiting the almost-diffeomorphisms g_n with the properties (a)–(e). These g_n are construct level by level, bottom up, using conformal identifications of flat surfaces with the same periods ([BCGGM3, Theorem 2.7]), a C^1-quasi-conformal extension of these maps across the plumbing cylinder and the equivalence of the conformal and C^1-quasi-conformal topology on strata of abelian differential ([BCGGM3, Section 2]).

To show that $\Omega\mathrm{Pl}$ is a homeomorphism we need to show that this map is open and locally injective. Openness amounts to showing that for any converging sequence in U^s, say converging to $(X, \omega, \widehat{\Gamma})$, we can eventually undo the plumbing construction and find $\Omega\mathrm{Pl}$-preimages in the model domain $\overline{\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})^s}$. These preimages are again found level by level, the scales t_i of the model differentials being determined by the scales s_i in the definition of convergence in U^s. Local injectivity amounts to checking uniqueness of the previous unplumbing steps using perturbed period coordinates. See [BCGGM3, Section 12.5-12.7] for details on these steps.

The action of $\mathbb{C}^*$ on the k-th root ω defines an action on the space $\Omega^k_+\mathcal{M}_{g,n}$ that is equivariant via $\lambda \mapsto \lambda^k$ with a $\mathbb{C}^*$-action on $\Omega^k\mathcal{M}_{g,n}$. The quotients of both actions is the same space $\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$. We encourage the reader to revisit all the steps to check that the first action extends equivariantly to all auxiliary spaces, multiplying simultaneously all forms at all levels. The resulting quotient of $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ by $\mathbb{C}^*$ is the compactification $\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ claimed in iii) of Theorem 1.1.

The *proof of Proposition 1.2* is contained in these statements, since Proposition 3.5 together with the disc coordinates x_j used in (16) gives local coordinates on $\overline{\mathfrak{W}^k_{\mathrm{pm}}(\widehat{\Gamma})^s} \times \Delta^h$. Consequently, the perturbed period coordinates are given by

$$\begin{aligned}
\mathrm{PPer}\colon \quad U^s \xrightarrow{\Omega\mathrm{Pl}^{-1}} W \times \Delta^h \qquad &\longrightarrow \mathbb{C}^h \times \mathbb{C}^{L+1} \times \prod_{i=0}^{-L} \mathbb{C}^{\dim E^{\mathrm{grc}}_{(i)} - 1} \\
[\widehat{X}, \boldsymbol{\omega}] \xmapsto{\Omega\mathrm{Pl}^{-1}} [(\widehat{X}, \boldsymbol{\eta}, \mathbf{t}, \mathbf{x})] \quad &\mapsto \Big(\mathbf{x}\,;\mathbf{t}\,;\prod_{i=0}^{-L} \mathrm{PPer}_i(\widehat{X}, \boldsymbol{\eta}, \mathbf{t})\Big)
\end{aligned} \tag{18}$$

on open orbifold charts U^s of $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$, using the inverse of the homeomorphism $\Omega\mathrm{Pl}$ constructed in Step 3 and 4.

4. The area form is good enough

Here we prove our main Theorem 1.4. We place ourselves in the setting of the theorem and recall that now $m_i > -k$ and thus the sets P and $\widehat{P}$ as defined in Section 2 are empty. The first step is to determine where the metric tends to infinity and then to give a convenient expression of the metric. Arguing inductively on k, we may also suppose that we are dealing with primitive k-differentials, i.e. that the canonical k-cover is connected.

We start with the definition of the corresponding hermitian form. For a symplectic basis $\alpha_1, \ldots, \alpha_{\widehat{g}}, \beta_1, \ldots, \beta_{\widehat{g}}$ of the absolute homology $H_1(\widehat{\Sigma}, \mathbb{Z})$ and for $\omega, \eta \in H^1(\widehat{X}, \mathbb{C})$ we define hermitian form

$$\langle \omega, \eta \rangle \;=\; \frac{i}{2} \sum_{i=1}^{\widehat{g}} (\omega(\alpha_i)\, \overline{\eta(\beta_i)} - \omega(\beta_i)\, \overline{\eta(\alpha_i)}) \tag{19}$$

with the abbreviations $\omega(\alpha_i) = \int_{\alpha_i} \omega$ etc. By Riemann's bilinear relations we can rewrite the metric defined in (1) using the hermitian form as

$$h(X, q) \;=\; \langle \omega, \omega \rangle \;=\; \frac{i}{2} \sum_i (a_i \overline{b_i} - b_i \overline{a_i})\,,$$

where we introduce another abbreviation $a_i = \omega(\alpha_i)$ and $b_i = \omega(\beta_i)$, to be used if ω is the only one-form that appears. We recall from (10) the notation $t_{\lceil i \rceil}$ for the product of the appropriate t_j-power of the levels above i.

Lemma 4.1. *The metric h extends to a smooth metric across boundary points with only vertical edges.*

Proof. A neighborhood of the point $(\widehat{X}, \boldsymbol{\omega})$ is also a neighborhood U of that point in the model domain. There, $\boldsymbol{\omega}$ is interpreted as a collection ω_i of non-zero differential forms on the subsurface $\widehat{X}_{(i)}$ on the i-th level. The neighborhood of $(\widehat{X}, \boldsymbol{\omega})$ consists of the stable differentials obtained via the plumbing construction applied to the differential forms $(\prod_{j=i}^{-1} t_j^{\ell_j})\eta_{(i)} + \xi_{(i)}$ on the universal family over model domain restricted to the small neighborhood. Here $\mathbf{t} = (t_i)_{i=-L}^{-1}$ is the collection of 'opening-up' parameters in the polydisc and the positive integers ℓ_j are determined by the enhanced level graph Γ via (6). The central fiber of this family agrees with $\boldsymbol{\omega}$ by construction. From the modification differentials $\xi = (\xi_{(i)})$ we mainly have to retain that they tend to zero faster than t^{ℓ_i}, see Definition 3.3 iii).

Consider the fiber $\widehat{\mathcal{X}}_{(i),u}$ over $u \in U$ of the level-i subsurface over the model domain. Let $E^+_{(i)}$ be the edges connecting that surface to higher levels and $E^-_{(i)}$ the edges connecting that surface to lower levels. Consider the subsurface where for each $e \in E^+_{(i)}$ the interior of the plumbing annuli $\mathcal{A}^-_e$ (and thus the pole) has been removed. The $\eta_{(i)}$-area of this subsurface is bounded for $u \in U$, since the areas of the plumbing cylinders $\mathbb{V}_e$ for $e \in E^-_{(i)}$ tend to the area of a disc with metric $z^m dz$ with $m \geq 0$. Consequently, for any sequence $(\widehat{X}_n, \omega_n)$ of surfaces plumbed from $(\widehat{X}_n, \boldsymbol{\eta}_n)$ using the parameters $\mathbf{t}_n$ with the property $t_i \to 0$ for all $i = -1, \ldots, -L$ and with $(\widehat{X}_n, \boldsymbol{\eta}_n) \to (\widehat{X}, \boldsymbol{\omega})$ in the model domain, we get

$$\operatorname{area}_{\Omega\mathrm{Pl}(\widehat{X}_n, \boldsymbol{\eta}_n, \mathbf{t}_n)}(\omega_n) \;=\; \sum_{i=0}^{-L} \operatorname{area}_{\widehat{X}_n}\big((\textstyle\prod_{j=i}^{-1} t_{j,n}^{\ell_j})\, \eta_{i,n} + \xi_{i,n}\big) \;\to\; \operatorname{area}_{\widehat{X}}(\omega_0)\,, \quad (20)$$

which is finite and non-zero. The smooth dependence on the parameters is obvious. $\square$

Now suppose we work in a neighborhood U of a general boundary point with notations $(\mathbf{x}\,; \mathbf{t}\,; \coprod_{i=0}^{-L} \mathrm{PPer}_i)$ for the perturbed period coordinates as in Proposition 1.2 and in detail in (18). More precisely, we group the vector $\mathbf{x}$ of coordinates for opening the horizontal nodes as $\mathbf{x} = (x_{(i),j})$, where (i) denotes the level that contains the node an where $j = 1, \ldots, n(i)$ labels these nodes.

Proposition 4.2. *On the neighborhood U the metric h has the form*

$$h(X, q) \;=\; \sum_{i=0}^{L} h_{(i)}, \quad h_{(i)} = |t_{\lceil i \rceil}|^2 \Big(c_{(i)} - \sum_j d_{(i),j} \log(|x_{(i),j}|^2) \Big)\,, \qquad (21)$$

where $c_{(i)}$ and $d_{(i),j}$ are smooth functions of the coordinates PPer_i and t_j that are bounded above and away from zero.

Proof. The total space of the line bundle $\mathcal{O}(-1)$ defined in the introduction as the φ-pullback of $\mathcal{O}(-1)$ from the incidence variety compactification, is nothing but the total space of the projection $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu) \to \mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$. Our goal is thus to find an expression for the area of a point in $\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$ near a boundary point $(\widehat{X}, \boldsymbol{\omega}, \widehat{\Gamma}) \in \partial\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu)$.

For notational simplicity we consider first the case **that X has only one horizontal node that we moreover suppose to be non-separating**. Consequently, $\widehat{X}$ has k nodes. We pick a convenient basis of $H_1(\widehat{\Sigma}, \mathbb{Z})$ on a smooth model $\widehat{\Sigma}$ (connected by our standing primitivity assumption) that is pinched to $\widehat{X}$. The k pinched curves $\alpha_i \in \widehat{\Sigma}$ are linearly independent and form a τ-orbit in homology. Next, we take the symplectic dual curves β_i with the intersection pairing $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$. Note that β_i is well-defined in a neighborhood of $(\widehat{X}, \boldsymbol{\omega}, \widehat{\Gamma})$ (only) up to adding an integer multiple of α_i. We arbitrarily complement these elements by $\alpha_i, \beta_i \in H_1(\widehat{\Sigma}, \mathbb{Z})$ for $i = k+1, \ldots, \widehat{g}$ to a symplectic basis.

In the current case the multi-scale differential case $\boldsymbol{\omega} = (\omega_0) = (\eta_0)$ consists of a single one-form. Recall from Step 2 in Section 3.7 that points in a neighborhood of $(\widehat{X}, \boldsymbol{\omega}, \widehat{\Gamma})$ are obtained from surfaces $(\widehat{X}', \eta') \in \Omega\mathcal{M}_{\widehat{g}-k, \widehat{n}+2k}(\widehat{\mu}, (-1)^{2k})$ that admit an action by $\langle \tau \rangle \cong \mathbb{Z}/k$, by gluing in k times each of the plumbing fixtures $\mathbb{W}$ in a τ-equivariant way, parameterized by a coordinate $\mathbf{x} = (x) \in \Delta$ as in Step 2 above. By Proposition 1.2 and explicitly (18) the coordinates near the boundary point are $\mathbf{x}$ and the periods in the ζ_k-eigenspace of η'. We denote by ω' the differential obtained from η' after the plumbing construction. Notice that ω' is a holomorphic differential on the plumbed surfaces having all plumbing parameters x_i different from zero. Our aim is to rewrite the area form, which is defined using ω' periods in the interior, in terms of the perturbed period coordinates, i.e. $\mathbf{x}$ and η' periods, which give charts near the boundary. We abbreviate $a_i = \omega'(\alpha_i)$ and $b_i = \omega'(\beta_i)$.

Next we decompose $\beta_j = \beta_j^X + \beta_j^\circ$ into the 'eXterior' part β_j^X outside the plumbing fixture and the part β_j° between the two seams of the plumbing fixture, as in Figure 1.

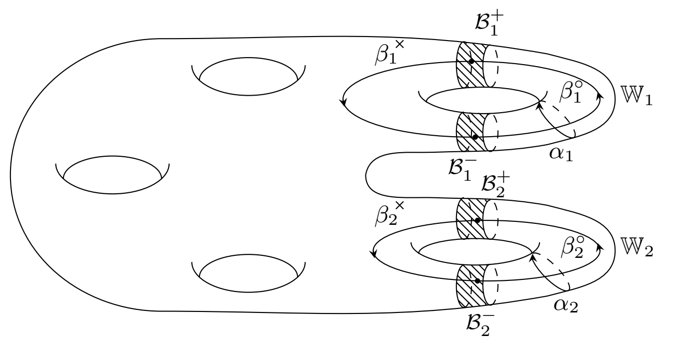

FIGURE 1. Decomposing the β_i into exterior and interior of the plumbing fixture

The separation happens at fixed sections (of the universal family over the stratum $\Omega\mathcal{M}_{g-kh,\widehat{n}+2kh}(\widehat{\mu},(-1)^{2kh})$) in the neighborhood of $(\widehat{X}',\eta')$ in the plumbing annuli $\mathcal{B}_j$, say at the points $u=\delta_0$ and $v=\delta_0$. Equation (17) simplifies in the one-level case to $\Omega_j = r_j dv/v$ where $r_j = \zeta^j a_1/2\pi i = a_j/2\pi i$. We compute

$$b_j \;=\; \int_{\beta_j}\omega' \;=\; \int_{\beta_j^X}\eta' + \int_{\delta_0}^{x/\delta_0}\Omega_j \;=\; \int_{\beta_j^X}\eta' + r_j(\log x - 2\log\delta_0)\,,$$

which is well-defined in $\mathbb{C}+r_j\mathbb{Z}$ because of the ambiguity of β_j. By definition of the area form and since ω' and η' agree outside the plumbing fixture,

$$\begin{aligned} h(X,q) \;&=\; \frac{i}{2}\sum_{j=1}^{k}(a_j\overline{b_j}-b_j\overline{a_j}) \;+\; \frac{i}{2}\sum_{j=k+1}^{\widehat{g}}(a_j\overline{b_j}-b_j\overline{a_j}) \\ &=\; C \;+\; \frac{i}{2}\sum_{j=k+1}^{\widehat{g}}(a_j\overline{b_j}-b_j\overline{a_j}) - \frac{k}{4\pi}\cdot|a_1|^2\log(|x|^2) \end{aligned} \tag{22}$$

is independent of the ambiguity of b_j. Here C is some function that stems from the integration in the thick part and that is independent of x. We may now let $c_{(0)} = C + \frac{i}{2}\sum_{j=k+1}^{\widehat{g}}(a_j\overline{b_j}-b_j\overline{a_j})$ and $d_{(0),1} = \frac{k}{4\pi}\cdot\pi|a_1|^2$. Both functions are smooth and bounded away from zero near $(\widehat{X},\boldsymbol{\omega},\widehat{\Gamma})$, in fact they correspond to the volume of the region outside the handles and the residue at the handle respectively.

For a **general X that has only horizontal nodes** we arrive at a similar formula. We decompose the plumbed surface of the canonical covering into the thick part and the plumbing fixtures $\mathbb{W}_j$, for $j=1,\ldots,n(0)$. Since the flat area is additive, we can write it as a sum of the contribution of the flat area of the thick part and the flat area of the $\mathbb{W}_j$, as we did in the previous case. The area of the thick part is clearly a smooth function of the period coordinates and bounded away from zero. For each node j of X, we get k-plumbing fixtures $\mathbb{W}_{j,l}$. Since the residues $r_{j,l}$ of the associated simple pole differential are τ-conjugates for each fixed j, they all have the same modulus that we denote by $|r_j|$. From the computation in the plumbing fixtures as in the previous case we see that the flat area is given by

$$h(X,q) \ = c_{(0)} - \frac{k}{4\pi}\sum_{j=1}^{n(0)}|r_j|^2\log(|x_j|^2) \tag{23}$$

which is of the shape we claimed.

In the **general case of X having of horizontal and vertical nodes**, we apply the previous procedure for each level. More precisely, recall that the plumbing construction decomposes the surface along the vertical edges into various levels (and the plumbing cylinders between the levels). For each level, we decompose the plumbed surface as the union of the horizontal plumbing fixtures, the vertical plumbing fixtures and the thick part. For the lowest level, the differential form is scaled with $t_{\lceil L\rceil}$ and the individual contribution of that level is as on the right hand side of (23), the total sum restricted to a basis of the homology of that level as explained in (2). This justifies the contribution for $h_{(L)}$. The area of the vertical plumbing fixtures scales with the upper end, so it does not contribute to $c_{(L)}$.

For a higher level i, we have the same structure, i.e. a contribution as on the right hand side of (23) scaled with $t_{\lceil i\rceil}$, except that now the flat area has to be calculated with respect to $t_{\lceil i\rceil}\eta_i + \xi_i$. This is relevant only when justifying that the constants $c_{(i)}$ (the area of the thick part of that level) and $d_{(i),j}$ (the ratio of the residue over the period used to fix the projective scale at that level) are bounded away from zero. This clearly holds at $\mathbf{t} = 0$ where there is no modification differential. It continues to hold in a neighborhood since the modification differential ξ_i scales with $t_{\lceil i+1\rceil}$, so after taking out a factor of $t_{\lceil i\rceil}$ we are still left with a smooth function. The area of the vertical plumbing fixtures connecting level i to the level $i-1$ contributes only in making the functions $c_{(i)}$ a little bit bigger. □

Before proceeding to the proof of Theorem 1.4 we recall as an aside and for comparison the definition of a good metric in the sense of [Mum77] on a smooth r-dimensional variety (or orbifold) $\overline{X}$.

Suppose that $\overline{X}$ is the compactification of X with a normal crossing boundary divisor $\partial X = \overline{X}\setminus X$. Let $\mathcal{L}$ be a line bundle on $\overline{X}$. A metric h on $\mathcal{L}|_X$ is *good*, if for each point $p \in \partial X$ there is a neighborhood Δ^r with coordinates such that $\partial X = \{\prod_{i=1}^k x_i = 0\}$ and such that the function $h_s = h(s,s)$ for a local generating section s of $\mathcal{L}$ has the following properties:

(i) There exist $C > 0$ and $n \in \mathbb{N}$ such that $|h_s| < C\Big(\sum_{i=1}^k \log|x_i|\Big)^{2n}$ and $|h_s^{-1}| < C\Big(\sum_{i=1}^k \log|x_i|\Big)^{2n}$.

(ii) the connection one-form $\partial\log h$ and the curvature two-form $\overline{\partial}\partial\log h$ have Poincaré growth.

Here a p-form η is said to have *Poincaré growth* on Δ^r if for any choice of sections v_i of $T_{\overline{X}}(\Delta^r)$ there is C such that

$$|\eta(v_1,\dots,v_p)|^2 \leq C\prod_{i=1}^p \omega_P(v_i,v_i)$$

holds for ω_P the product of the Poincaré metrics $|dx_i|^2/x_i^2\log|x_i|^2$ in the coordinates x_i for $i \leq k$ and the euclidean metric in the other coordinates.

Mumford shows ([Mum77, Theorem 1.4]) that for a good metric h the curvature form $\frac{i}{2\pi}[F_h]$ defines a closed $(1,1)$-current that represents the first Chern class of $\mathcal{L}$. This estimate boils down to the observation that the 'Poincaré' integral

$$\frac{1}{2\pi i}\int_{\Delta_\varepsilon}\frac{dxd\bar{x}}{|x|^2(\log|x|^2)^2} \;=\; -\int_0^\varepsilon \frac{ds}{s(\log(s))^2} \;=\; \frac{1}{\log\varepsilon} < \infty. \tag{24}$$

of the Poincaré metric on the (punctured) disc Δ_ε is finite and goes to zero as $\varepsilon \to 0$.

The metric h is indeed good if there is only one level, i.e. if the graph has no vertical edges, as one can deduce from the estimates in the propositions below. However *the metric h fails to be good* if there are several levels and horizontal nodes on lower level. Consider the simplest such case of a graph with two levels, one vertex at each level and two edges, one edge joining the levels and a horizontal edge on lower level. Simplifying the situation by assuming that the bounded functions $c_{(i)}$ and $d_{(i)}$ are 1, the metric is then given by

$$h(X,q) \;=\; 1 + |t_1|^2(1-\log(|x|^2)\,.$$

We observe that this metric is not good in the sense of Mumford near the point $(t_1,x) = (0,0)$, neither considering the natural boundary $\{x = 0\}$ (Case 1) consisting of the complement of the locus where the metric smoothly extends, nor if we try to incorporate the problematic t_1-direction into the boundary and try the boundary $\{x = 0\} \cup \{t_1 = 0\}$ (Case 2).

Suppose the metric were good and we are in Case 2. Then we would have a constant C such that

$$\Big|\partial \log h\Big(\frac{\partial}{\partial t_1}\Big)\Big|^2 \leq \frac{C}{|t_1|^2(\log(|t_1|^2))^2}$$

on the neighborhood U of $(t_1,x) = (0,0)$, which is equivalent to the inequality of the square roots

$$\frac{|t_1|(1-\log(|x|^2)}{1+|t_1|^2(1-\log(|x|^2))} \leq \frac{C^{1/2}}{|t_1|(\log(|t_1|^2))} \tag{25}$$

Choosing a sequence tending to $(0,0)$ with $1-\log(|x|^2) = |t_1|^{-2}$ we get a contradiction. In Case 1 we arrive at equation (25) without the denominator of the right and side and the same special sequence yields a contradiction.

Instead of aiming for a bound as in the definition of good, integrability statements are sufficient. In fact the coefficients of

$$\partial\log(h) = \frac{|t_1|^2(1-\log(|x|^2)}{h}\frac{dt_1}{t_1} - \frac{|t_1|^2}{h}\frac{dx}{x}$$

and

$$\overline{\partial}\partial\log h \;=\; \frac{|t_1|^2(1-\log|x|^2)}{h^2}\frac{d\bar{t}_1}{\bar{t}_1}\frac{dt_1}{t_1} - \frac{|t_1|^4}{h^2}\frac{d\bar{x}}{\bar{x}}\frac{dx}{x} - \frac{|t_1|^2}{h^2}\frac{d\bar{t}_1}{\bar{t}_1}\frac{dx}{x} - \frac{|t_1|^2}{h^2}\frac{d\bar{x}}{\bar{x}}\frac{dt_1}{t_1}$$

are locally integrable, and thus define currents. To see that the current $F_h = [\overline{\partial}\partial\log h]$ given by the curvature form is closed, we have to show that we can apply the derivative (in the sense of currents) inside the brackets, on the differential form, where it gives zero. This requires an application of Stokes' theorem, and thus an integral over the boundary T_δ of a shrinking tubular neighborhood around the locus where the metric is not smooth, i.e. around $x = 0$. To see that this current represents the first chern class $c_1(\mathcal{O}(-1))$, we compare with the curvature form of a smooth metric. To see that the difference is zero in cohomology, the term $[d\partial \log(h)]$ appears and we'd like to invoke say that this is $d[\partial \log(h)]$, i.e. a coboundary of a current. This is gives a second application of Stokes' theorem, justified by another integration over T_δ. To justify that we can pass to wedge powers is a third application of Stokes' theorem. This integrals are estimated in the general case in the following proofs, see in particular (41) and also (40) for an additional case that appears when more levels are present.

The *Proof of Theorem 1.4* is now contained in the following two propositions.

Proposition 4.3. *The differential forms $\Omega = \partial \log h$ and $F_h = \overline{\partial}\partial \log h$, and more generally the forms F_h^d and $\Omega \wedge F_h^d$ have coefficients in L^1_{loc}.*

In particular F_h^d defines a current of type (d, d) for any $d \in \mathbb{N}$.

Proposition 4.4. *The current $[\overline{\partial}\partial \log h]$ is closed and $\frac{1}{2\pi i}$ times the curvature $(1,1)$-form $F_h = \overline{\partial}\partial \log h$ represents the first Chern class $c_1(\mathcal{O}(-1))$ in cohomology.*

More generally, the wedge powers $\wedge^d \left(\frac{1}{2\pi i} F_h\right)$ represent the class of $c_1(\mathcal{O}(-1))^d$ in cohomology.

We first calculate the differential forms explicitly. First $\partial \log(h) = \frac{1}{h}\sum_{i=0}^{L} \partial h_{(i)}$ where

$$\partial h_{(i)} \;=\; \sum_{k=1}^{i} \ell_k h_{(i)} \frac{dt_k}{t_k} + |t_{\lceil i\rceil}|^2 \left(\partial c_{(i)} - \sum_{j=1}^{n(i)} d_{(i),j}\, \frac{dx_{(i),j}}{x_{(i),j}} + \log(|x_{(i),j}|^2)\, \partial d_{(i),j} \right) \tag{26}$$

where we note that $\partial c_{(i)}$ and $\partial d_{(i),j}$ involves only the differentials of the coordinates in PPer_i. For the computation of F_h we need

$$\begin{aligned}
\overline{\partial}\partial h_{(i)} \;=\; & \sum_{k_1,k_2=1}^{i} \ell_{k_1}\ell_{k_2} h_{(i)} \frac{d\bar{t}_{k_1}}{\bar{t}_{k_1}} \frac{dt_{k_2}}{t_{k_2}} - \sum_{j=1}^{n(i)}\sum_{k=1}^{i} \ell_k d_{(i),j} |t_{\lceil i\rceil}|^2 \frac{d\bar{x}_{(i),j}}{\bar{x}_{(i),j}} \frac{dt_k}{t_k} \\
& - \sum_{j=1}^{n(i)}\sum_{k=1}^{i} \ell_k d_{(i),j} |t_{\lceil i\rceil}|^2 \frac{d\bar{t}_k}{\bar{t}_k} \frac{dx_{(i),j}}{x_{(i),j}} \\
& + \sum_{k=1}^{i} \ell_k |t_{\lceil i\rceil}|^2 \Big(\overline{\partial} c_{(i)} - \sum_{j=1}^{n(i)} \log(|x_{(i),j}|^2) \overline{\partial} d_{(i),j} \Big) \wedge \frac{dt_k}{t_k} \\
& + \sum_{k=1}^{i} \ell_k \sum_{j=1}^{n(i)} |t_{\lceil i\rceil}|^2 \frac{d\bar{t}_k}{\bar{t}_k} \wedge \Big(\partial c_{(i)} - \sum_{j=1}^{n(i)} \log(|x_{(i),j}|^2) \partial d_{(i),j} \Big) \\
& - \sum_{j=1}^{n(i)} |t_{\lceil i\rceil}|^2 \Big(\overline{\partial} d_{(i),j} \wedge \frac{dx_{(i),j}}{x_{(i),j}} + \frac{d\bar{x}_{(i),j}}{\bar{x}_{(i),j}} \wedge \partial d_{(i),j} \Big) \\
& + |t_{\lceil i\rceil}|^2 \Big(\overline{\partial}\partial c_{(i)} - \sum_{j=1}^{n(i)} (\log(|x_{(i),j}|^2) \overline{\partial}\partial d_{(i),j} \Big)
\end{aligned} \tag{27}$$

Finally,

$$F_h = \overline{\partial}\partial \log h = \frac{\overline{\partial}\partial h}{h} - \frac{\overline{\partial} h}{h} \wedge \frac{\partial h}{h}$$
$$= \sum_{i=0}^{L} \frac{\overline{\partial}\partial h_{(i)}}{h} - \sum_{i_1=0}^{L}\sum_{i_2=0}^{L} \frac{\overline{\partial} h_{(i_1)}}{h} \wedge \frac{\partial h_{(i_2)}}{h} \tag{28}$$

The main information to retain from these formulas is that $\partial \log(h)$ and $\overline{\partial}\partial \log(h)$ consists of linear combinations of the building blocks

$$\frac{h_{(i)}}{h}\frac{dt_k}{t_k}, \quad \frac{|t_{\lceil i \rceil}|^2}{h}\frac{dx_{(i),j}}{x_{(i),j}}, \quad \partial c_{(i)}, \quad \partial d_{(i),j} \tag{29}$$

with smooth functions as coefficients, resp. of two-fold wedge products of type $(1,1)$ of these building blocks and the version with a bar everywhere together with the building blocks

$$\frac{h_{(i)}}{h}\frac{d\bar{t}_{k_1}}{\bar{t}_{k_1}}\frac{dt_{k_2}}{t_{k_2}}, \quad \frac{|t_{\lceil i \rceil}|^2}{h}\frac{d\bar{x}_{(i),j}}{\bar{x}_{(i),j}}\frac{dt_k}{t_k}, \quad \frac{|t_{\lceil i \rceil}|^2}{h}\frac{d\bar{t}_k}{\bar{t}_k}\frac{dx_{(i),j}}{x_{(i),j}}, \quad \overline{\partial}\partial c_{(i)} \quad \text{and} \quad \overline{\partial}\partial d_{(i),j}\,. \tag{30}$$

for $k, k_1, k_2 \geq i$.

We fix some more notation. We may assume that our neighborhood U is the product of the polydiscs

$$D^{\mathbf{t}} = \{\mathbf{t} : t_{(i)} \in \Delta_\varepsilon\} \quad \text{and} \quad D^{\mathbf{x}} = \{\mathbf{x} : x_{(i),j} \in \Delta_\varepsilon\}\,. \tag{31}$$

in the corresponding variables times a ball B corresponding to all the variables in PPer_i for $i = 0, \ldots, -L$. We use the change $x_{(i),j} = s_{(i),j}e^{\sqrt{-1}\vartheta_{(i),j}}$ to polar coordinates throughout.

Proof of Proposition 4.3. We examine each term in $\partial \log h$. For dt_k-terms we use $h \geq c_{(0)}$ and that $|t_{\lceil i \rceil}|^2/t_k$ is a polynomial expression in the t_i and $\bar{t}_i$. The logarithmic contribution in $h_{(i)}$ is unbounded, but after bounding the contribution from the t-variables, estimating $c_{(i)}$ and $d_{(i),j}$ from above and change to polar coordinates we are left with

$$\int_{B\times D^{\mathbf{x}}\times D^{\mathbf{t}}} \frac{h_{(i)}}{ht_k}d\mathrm{vol} \leq C_1 \int_{D^{\mathbf{x}}} c_{(i)} - 2d_{(i),j}\sum_{j=1}^{n(i)} \log(|x_{(i),j}|) \prod_j |dx_{(i),j}|^2$$
$$\leq C_2 \int \cdots \int_{\{s_{(i),j}\leq \varepsilon\}} \Big(1 - \sum_{j=1}^{n(i)} \log s_{(i),j}\Big) \prod_{j=1}^{n(i)} s_{(i),j}ds_{(i),j} < \infty\,. \tag{32}$$

For the $dx_{(i),j}$-coefficients we use $h \geq c_{(0)}$ and we are left with a polynomial expression in the t_i and $\bar{t}_i$ and the finite integral $\int_{\Delta_\varepsilon} \frac{1}{x_{(i),j}}|dx_{(i),j}|^2$. The same arguments apply verbatim to the coefficients of $\overline{\partial}\log h$.

Next, we examine the coefficients $\frac{1}{h}\overline{\partial}\partial h_{(i)}$, which can be treated by the previous arguments. For $d\bar{t}_{k_1}dt_{k_2}$ note that $\bar{t}_{k_1}t_{k_2}$ divides $h_{(i)}$ for $k_j \geq i$ and for the mixed terms like $d\bar{x}_{(i),j}dt_{k_2}$ the combination of that divisibility argument and the integrability in (32) suffices. The remaining terms are bounded, since the logarithmic terms are governed by h in the denominator.

Finally we examine the terms that may arise from as an arbitrary wedge product of $\frac{1}{h}\overline{\partial}\partial h_{(i)}$, of $\frac{1}{h}\partial h_{(i)}$ or of $\frac{1}{h}\overline{\partial}h_{(i)}$. We note that we can use the denominator $h \geq h_{(i)}$ so that each $dx_{(i),j}$ appears with coefficient $1/x_{(i),j}\log(|x_{(i),j}|)$ and $d\bar{x}_{(i),j}$

appears with coefficient $1/\bar{x}_{(i),j}\log(|x_{(i),j}|)$, and so that each $\log(|x_{(i),j}|^2)$-prefactor has $h_{(i)}$ as denominator. Consequently, the integrals that appear involve bounded functions of $x_{(i),j}$, polynomials in the t_i and $\bar{t}_i$, the integral of $1/x_{(i),j}\log(|x_{(i),j}|)$ and the Poincaré integral (24), all of which are finite.

For those differential forms that are globally well-defined, independent of the local scale of the metric being, in L^1_{loc} is sufficient to define a current. $\square$

Proof of Proposition 4.4. We identify the local statements needed to prove the claims and justify them simultaneously. To see that $[F_h] = [\overline{\partial}\partial\log h]$ defines a closed current we need to justify the first step in the chain

$$d[\overline{\partial}\partial\log h] \;=\; [d(\overline{\partial}\partial\log h)] \;=\; 0 \tag{33}$$

of cohomology classes of currents. By definition we have to justify that

$$\int_{D\times B} dF_h\wedge\xi = -\int_{D\times B} F_h\wedge d\xi \tag{34}$$

for any smooth r-form ζ, where $r = \dim_{\mathbb{R}}\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu) - 3$. By Stokes' theorem amounts to justify that for all (i,j) the limit

$$\lim_{\delta\to 0}\int_{T_\delta^{(i),j}} F_h\wedge\xi \;=\; 0\,. \tag{35}$$

where

$$T_\delta^{(i),j} \;=\; B\times\Big\{|x_{(i),j}|{=}\,\delta\,;\; t_{(k')}\in\Delta_\varepsilon \text{ for all } k'\,;\; x_{(i'),j'}\in\Delta_\varepsilon \text{ for all } (i',j')\neq(i,j)\Big\}$$

is the boundary of a tubular neighborhood inside $B\times D^{\mathbf{t}}\times D^{\mathbf{x}}$ around the divisor $x_{(i),j}=0$ with tube radius δ. Note that we do not need to consider the boundary of a tubular neighborhood around $t_i = 0$, since the metric extends smoothly across these loci by Lemma 4.1.

For the second statement let h^* be a smooth metric on $\mathcal{O}(-1)$. Then certainly $\frac{1}{2\pi i}$ times the curvature $F_h^* = \overline{\partial}\partial\log h^*$ represents the first Chern class of $\mathcal{O}(-1)$. To justify the equality of cohomology classes of currents

$$[\overline{\partial}\partial\log h^*] - [\overline{\partial}\partial\log h] \;=\; [d(\partial\log h^* - \partial\log h)] \;=\; d[\partial\log h^* - \partial\log h] \;=\; 0$$

we have to justify the interchange of the derivative and passing to the current in the second equality sign. Then the last equality follows from Proposition 4.3, showing that the expression is a coboundary in the sense of currents, since $\log h^* - \log h = \log(h^*/h)$ is independent of the scale of h and thus globally well-defined.

Writing $\Omega^* = \partial\log h^*$ and $\Omega = \partial\log h$, we have to justify that for any smooth $\dim_{\mathbb{R}}\mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu) - 2$-form and for all (i,j)

$$\lim_{\delta\to 0}\int_{T_\delta^{(i),j}}(\Omega^*-\Omega)\wedge\xi \;=\; 0\,, \quad \text{which follows from} \quad \lim_{\delta\to 0}\int_{T_\delta^{(i),j}}\Omega\wedge\xi \;=\; 0 \tag{36}$$

and from the smoothness of Ω^*.

For the generalization to wedge powers we use $F_h = d\Omega$ and $F_h^* = d\Omega^*$ and want to argue that there is an equality of cohomology classes of currents

$$[F_h^d] - [(F_h^*)^d] \;=\; d\Big[(\Omega-\Omega^*)\wedge\sum_{i+j=n-1}F_h^i(F_h^*)^j\Big]\,.$$

With this equation at hand we use that the argument of the differential operator on the right hand side defines a current by Proposition 4.3, so that $[F_h^d]$ and $[(F_h^*)^d]$ are cohomologous and $(\frac{1}{2\pi})^d[(F_h^*)^d]$ is known to represent $c_1(\mathcal{O}(-1))^d$.

To justify this equation we need to argue that for all (i,j), for all d and for all smooth $\dim_{\mathbb{R}} \mathbb{P}\Xi^k\overline{\mathcal{M}}_{g,n}(\mu) - 2d - 1$-forms ξ the hypothesis

$$\lim_{\delta\to 0} \int_{T_\delta^{(i),j}} \Omega \wedge F_h^n \wedge \xi \;=\; 0 \tag{37}$$

for the interchange of derivative and passage to the current holds.

To justify these three equations (35), (36) and (37) we fix the tubular neighborhood $T_\delta^{(i),j}$ around one of the boundary divisors and analyze the forms that may appear from the wedge products in (35), in (36) or in (37). These are wedge products of the building blocks in (29) and (30) and the differentials of the coordinates themselves.

Our strategy is to apply Fubini's theorem for the different variables, discuss the contributions of the building blocks and identify which estimates of the other building blocks are sufficient to ensure that the integration of the variable under consideration provides a finite result, independent of δ, and going to zero for the variables $x_{(i),j}$ and $\bar{x}_{(i),j}$.

We start with the terms contributing to $dx_{(i'),j'}d\bar{x}_{(i'),j'}$ for $(i',j') \neq (i,j)$. The building blocks involving these differential forms give coefficients that are either smooth or they are smooth multiples of

$$\frac{|t_{\lceil i'\rceil}|^2}{hx_{(i'),j'}} \quad \text{or} \quad \frac{|t_{\lceil i'\rceil}|^2}{h\bar{x}_{(i'),j'}} \quad \text{or} \quad \frac{|t_{\lceil i'\rceil}|^4}{h^2|x_{(i'),j'}|^2}\,. \tag{38}$$

We will make two types of estimates from them, 'in isolation' or 'in combination with t-variables'. When considered in isolation, we will make sure in these displayed cases that the coefficients of the other building blocks will have no other contribution involving the variable $x_{(i'),j'}$. Then we can integrate these variables and obtain a finite result independently of δ: In the last case we use the estimate $h \geq h_{(i')}$, that implies

$$\frac{t_{\lceil i'\rceil}|^4}{h^2} \leq \frac{1}{(c_{(i')} - d_{(i'),j'}\log(|x_{(i'),j'}|^2)^2}$$

and the finiteness of the Poincaré integral. In the first two cases we do not rely on the presence of the $|t_{\lceil i'\rceil}|^2/h$ prefactor, the bound $h \geq c_{(0)}$ is sufficient. If the $dx_{(i'),j'}d\bar{x}_{(i'),j'}$ building blocks appears with smooth coefficients we will make sure that the other building blocks give at most as powers $\log(|x_{(i'),j'}|^2)$ and obtain a finite integral with the same argument that as in the last line of (32).

Next we analyze the building blocks involving dt_k and $d\bar{t}_k$. These are smooth (possibly also stemming from derivatives of $c_{(i)}$ or $d_{(i),j}$) or involve

$$\frac{h_{i'}}{ht_k} \quad \text{or} \quad \frac{h_{i'}}{h\bar{t}_k}, \quad (|i'| \geq |k|)\,. \tag{39}$$

We focus on the level k. If there are no horizontal nodes on that level we use that $h_k = |t_{\lceil k\rceil}|^2$ and $h \geq c_{(0)}$ to cancel the denominator and obtain a finite integral. This also works in the smooth case. Suppose there is a horizontal node at this level, say corresponding to $x_{(k),j'}$ with (k,j') being different from the distinguished pair (i,j). For this variable, we cannot treat the prefactors from (38) in isolation, but have to estimate jointly ('in combination') with that of dt_k and $d\bar{t}_k$. In the first two cases we can use $|t_{\lceil i'\rceil}|^2$ to cancel the t_k and $\bar{t}_k$ in the denominator. The integral is now finite. The last case is the crucial one, combined with $\frac{h_{(i')}}{h}\frac{dt_k}{t_k}$ and its complex conjugate. (Combinations of the building blocks in (30) are treated with the same estimates.) Now we estimate $h \geq h_{(i')}$ and $|t_{\lceil i'\rceil}|^2 \leq |t_{i'}|^2$ to use part of the numerator from (30) to take care of the $t_{i'}$-denominators. Passing to polar coordinates $t_k = re^{\sqrt{-1}\theta}$ and $x_{(k),j'} = se^{\sqrt{-1}\vartheta}$ it now suffices to observe that (keeping the second and third term of the binomial expansion on the passage from the first to the second line, using the substitution $u = r(1-\log(s^2))^{1/2}$ and then $v = 1-\log(s)$)

$$\begin{aligned}
&\int_{\Delta_\varepsilon^2} \frac{|t_{\lceil i'\rceil}|^2}{h^2|x_{(k),j'}|^2}|dx_{(k),j'}|^2|dt_{i'}|^2 \leq C_1\int_0^\varepsilon\int_0^\varepsilon \frac{r^3drds}{s(1+r^2(1-\log(s^2)))^2} \qquad (40)\\
\leq\ & C_2\int_0^\varepsilon\int_0^\varepsilon \frac{rdrds}{s(1-\log(s^2))(2+r^2(1-\log(s^2)))}\\
=\ & C_2\int_0^\varepsilon \frac{ds}{s(1-2\log(s))^2}\int_0^{\varepsilon(1-\log(s^2))^{1/2}} \frac{udu}{2+u^2}\\
=\ & \frac{C_2}{2}\int_0^\varepsilon \frac{\log(2+\varepsilon^2(1-2\log(s)))\,ds}{s(1-\log(s^2))^2} \leq C_3\int_0^\varepsilon \frac{-\log(1-\log(s))ds}{s(1-\log(s))^2}\\
=\ & C_3\int_{e^{1-\varepsilon}}^\infty \frac{\log(v)dv}{v^2} < \infty
\end{aligned}$$

where the constant C_1 appears from estimating $c_{(i')}$ and $d_{(i'),j}$ and from the angular integration. If there is more than one horizontal node at this level $k = i' = 1$, we treat the remaining ones 'in isolation' as stated in the previous paragraph.

The same argument works for the other levels, except that we have to treat the case $k = i$ specifically. We may focus on the variable $x_{(i),j}$ and t_i, treating the other variables $x_{(i'),j}$ (with their conjugates) 'in isolation'. Obviously the differential forms involving $dx_{(i),j}d\bar{x}_{(i),j} = s_{(i),j}ds_{(i),j}d\theta_{(i),j}$ restricts to zero on $T_\delta^{(i),j}$. Consequently at most one of the building blocks involving $dx_{(i),j}$ or $d\bar{x}_{(i),j}$ may appear. We use again polar coordinates $t_i = re^{\sqrt{-1}\theta}$ and $x_{(i),j} = \delta e^{\sqrt{-1}\vartheta}$. The crucial case is the estimate

$$\begin{aligned}
&\int_{\partial\Delta_\delta^{x_{(i),j}}}\int_{\Delta_\varepsilon^{t_i}} \frac{h_{(i)}^2}{h^2}\frac{|t_{\lceil i\rceil}|^2}{h}\frac{dt_i}{t_i}\frac{d\bar{t}_i}{\bar{t}_i}\frac{dx_{(i),j}}{x_{(i),j}} \leq C\int_0^\varepsilon \frac{rdr}{1+r^2(1-\log(\delta))}\\
=\ & \frac{C}{(1-\log(\delta))}\int_0^{\varepsilon(1-\log(\delta))^{1/2}} \frac{udu}{(1+u^2)} \qquad (41)\\
=\ & \frac{C}{(1-\log(\delta))}\cdot\frac{1}{2}\cdot\Big(\log(1+\varepsilon^2(1-\log(\delta)))\Big)
\end{aligned}$$

where we used $h \geq \tilde{C}(1+|t_i|^2(1-\log(|x_{(i),j}|)))$ with $h_{(i)}^2 \leq h^2$ and $|t_{\lceil i\rceil}|^2 \leq |t_i|^2$ in the first inequality, where we substituted $u = r(1-\log(s^2))^{1/2}$, and where again the constant C appears from estimating $c_{(i)}$ and $d_{(i),j}$ and from the angular integration. This expression tends to zero as $\delta \to 0$, as requested.

Finally we integrate the variables from PPer_i that appear in the $c_{(i)}$ and $d_{(i),j}$ which gives bounds independently of δ, since these are smooth differential forms. $\square$

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